Systems of Linear Equations

Note Title 9/2/2011

We begin with the following question:

Can you find x & y that satisfy

$$\begin{cases} 2x+3y=7 \\ 2x+2y=6 \end{cases}$$

(One possible method would be to solve for x in one equation, then substitute it in the other.) But instead, we will "subtract" one equation from the other:

$$\begin{cases} 2x+3y=7 \\ 2x+2y=6 \end{cases} \Rightarrow \begin{cases} 2x+3y=7 \\ y=1 \end{cases}$$

$$\Rightarrow \begin{cases} y=1 \\ x=2 \end{cases}$$

Questions?

1. Is there always a solution?
2. If so, is there always only one solution?
3. What if we have more that 2 variables?

Can we find a method of solving this kind of problem that answers all of these questions at once?

Yes, Gauss Elimination (i.e., Chapter 1)

Gauss Elimination uses Row Operations to find solutions to these kinds of problems

First, let's look at 2 more examples

(A) $\begin{cases} 3x+4y=5 \\ 6x+8y=10 \end{cases}$ Infinitely many solutions

(B) $\begin{cases} 3x+4y=5 \\ 6x+8y=9 \end{cases} \Rightarrow \begin{cases} 6x+8y=10 \\ 6x+8y=9 \end{cases} \Rightarrow \begin{cases} 6x+8y=10 \\ 0=1 \end{cases}$ NO Solutions

Gauss Elimination

This will give us a careful, systematic method to solve systems of linear equations, as well as to know when there are no solutions

① $\leftrightarrow$
② $\lambda A \rightarrow A$
③ $A + \lambda B \rightarrow A$

Ex Suppose we are given

$$\begin{cases} x+y+z=6 \\ 2x+3y-z=5 \\ -x+y+2z=7 \end{cases}$$

We are allowed to manipulate this system using any of the 3 types of Row Operations

1. Interchange two Rows
2. Multiply any one of the equations by any real number except 0
3. Add a multiple of one Row to another

We will use these to put our system into a standard form called Row Echelon Form

$$\begin{aligned} x+y+z&=6 \\ 2x+3y-z&=5 \\ -x+y+2z&=7 \end{aligned} \qquad R_3+R_1 \rightarrow R_3$$

$$\begin{aligned} x+y+z&=6 \\ 2x+3y-z&=5 \\ 0+2y+3z&=13 \end{aligned} \qquad R_2-2R_1 \rightarrow R_2$$

$$\begin{aligned} x+y+z&=6 \\ y-3z&=-7 \\ 2y+3z&=13 \end{aligned} \qquad \begin{aligned} &\cancel{-2R_2+R_3 \rightarrow R_2} \\ &R_3-2R_2 \rightarrow R_3 \end{aligned}$$

$$\begin{aligned} x+y+z&=6 \\ y-3z&=-7 \\ 9z&=27 \end{aligned} \qquad \tfrac{1}{9}R_3 \rightarrow R_3$$

$$\begin{aligned} x+y+z&=6 \\ y-3z&=-7 \\ z&=3 \end{aligned}$$

This is in <u>Row Echelon Form</u>

Now we "backsolve"

$$x + y + z = 6$$
$$y - 3z = -7$$
$$z = 3$$

Row 3: $z = 3$

Row 2: $y - 9 = -7 \Rightarrow y = 2$

Row 1: $x + 2 + 3 = 6 \Rightarrow x = 1$

However, we will use different notation!

Ex
$$x_1 + x_2 + x_3 = 0$$
$$x_1 + 2x_2 + 3x_3 = -1$$
$$2x_1 - 3x_2 + 4x_3 = 5$$

First, we write this as an "augmented matrix"

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & -1 \\ 2 & -3 & 4 & 5 \end{array}\right) \quad R_3 - 2R_2 \to R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & -1 \\ 0 & -7 & -2 & 7 \end{array}\right) \quad R_2 - R_1 \to R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & -7 & -2 & 7 \end{array}\right) \quad R_3 + 7R_2 \to R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 12 & 0 \end{array}\right) \quad \frac{1}{12} R_3 \to R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Defn An augmented matrix is in Row Echelon Form if
1. All of the all-zero rows are at the bottom
2. All of the row begin with some or no zeros followed by a 1. These 1's are called pivot entries.
3. Any pivot entry lower than another is also to the right of it.

Continuing with back solving:

Row 3: $x_3 = 0$

Row 2: $x_2 + 2(0) = -1 \Rightarrow x_2 = -1$

Row 1: $x_1 - 1 + 0 = 0 \Rightarrow x_1 = 1$

We can incorporate backsolving into Gauss Elimination by putting the augmented matrix in Reduced Row Echelon Form

Defn RREF is the same as above, but also

4. If a column has a pivot entry, then every other entry in the column is zero.

Continuing our example

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad R_1 - R_3 \to R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad R_2 - 2R_3 \to R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad R_1 - R_2 \to R_1$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The solution!!

Let's see how Gauss Elimination deals with the cases of no solutions + more than one solution:

Example 7

$$x_1 + x_2 + x_3 = 3$$
$$x_1 + 2x_2 + 3x_3 = 5$$
$$x_1 + 3x_2 + 5x_3 = 6$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 5 \\ 1 & 3 & 5 & 6 \end{array}\right) \quad R_3 - R_1 \to R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 5 \\ 0 & 2 & 4 & 3 \end{array}\right) \quad R_2 - R_1 \to R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 4 & 3 \end{array}\right) \quad R_3 - 2R_2 \to R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

Here, if we backsolve, Row 3 says $0 = -1$, so there are no solutions

Example 8

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 5 \\ 1 & 3 & 5 & 7 \end{array}\right) \quad R_3 - R_2 \to R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 5 \\ 0 & 1 & 2 & 2 \end{array}\right) \quad R_2 - R_1 \to R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right) \quad R_3 - R_2 \to R_3$$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

This is in REF

Backsolving

First Let $x_3 = s$

Row 2 $\quad x_2 + 2s = 2 \Rightarrow x_2 = 2 - 2s$

Row 1 $\quad x_1 + (2 - 2s) + s = 3$

$\quad x_1 = 1 + s$

The General solution is

$$(x_1, x_2, x_3) = (1 + s, 2 - 2s, s)$$

Example 17

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & -3 & 4 & 1 \\ 2 & -5 & -3 & 12 & 6 & 2 \\ 7 & 1 & 8 & 5 & 3 & 7 \end{array}\right) \quad R_2 - 2R_1 \to R_2$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & -3 & 4 & 1 \\ 0 & -9 & -9 & 18 & -2 & 0 \\ 7 & 1 & 8 & 5 & 3 & 7 \end{array}\right) \quad R_3 - 7R_1 \to R_3$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & -3 & 4 & 1 \\ 0 & -9 & -9 & 18 & -2 & 0 \\ 0 & -13 & -13 & 26 & -25 & 0 \end{array}\right) \quad \begin{array}{l} -\frac{1}{9}R_2 \to R_2 \\ -\frac{1}{13}R_3 \to R_3 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & -3 & 4 & 1 \\ 0 & 1 & 1 & -2 & \frac{2}{9} & 0 \\ 0 & 1 & 1 & -2 & \frac{25}{13} & 0 \end{array}\right) \quad R_3 - R_2 \to R_3$$

$$\begin{array}{c} \left(\begin{array}{ccccc|c} 1 & 2 & 3 & -3 & 4 & 1 \\ 0 & 1 & 1 & -2 & \frac{2}{9} & 0 \\ 0 & 0 & 0 & 0 & \frac{25}{13} - \frac{2}{9} & 0 \end{array}\right) \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

Back solving:

Row 3: $x_5 = 0$

Let $x_4 = s$

Let $x_3 = t$

Row 2: $x_2 + t - 2s + 0 = 0$

$x_2 = 2s - t$

Row 1: $x_1 + 2(2s - t) + 3t - 3s \overset{+0}{\wedge} = 1$

$x_1 + 4s - 2t + 3t - 3s = 1$

$x_1 = 1 - s - t$

General solution

$$(x_1, x_2, x_3, x_4, x_5) = (1-s-t,\ 2s-t,\ t,\ s,\ 0)$$

Homework For Thursday Chapter 1: 1, 3, 4, 6b, 7