Matrix multiplication

We do like this

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

or $$(a\ b\ c\ d\ e)\begin{pmatrix} f \\ g \\ h \\ i \\ j \end{pmatrix} = (af+bg+ch+di+ej)$$

More generally if A is an $m \times n$ matrix and if B is an $n \times p$ matrix

where $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ $B = \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix}$

then $AB = \begin{pmatrix} q_{11} & \cdots & q_{1p} \\ \vdots & & \\ q_{m1} & \cdots & q_{mp} \end{pmatrix}$

where $q_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$

Example $A = \begin{pmatrix} 3 & 7 \\ 2 & 4 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 5 & 4 & 0 & 2 \\ 9 & -1 & 1 & \pi \end{pmatrix}$

$$AB = \begin{pmatrix} 78 & 5 & 7 & 6+7\pi \\ 46 & 4 & 4 & 4+4\pi \\ 9 & -1 & 1 & \pi \end{pmatrix}$$

Caution! Matrix multiplication is only defined if

\# col in left matrix = \# rows in right matrix

Caution! Matrix multiplication is not commutative

Ex

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 19 & -1 \\ 0.5 & 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 19.5 & 1 \\ 19.5 & 1 \end{pmatrix} \qquad BA = \begin{pmatrix} 18 & 18 \\ 2.5 & 2.5 \end{pmatrix}$$

Motivation for this definition of matrix mult?

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 \dots + a_{mn}x_n &= b_m \end{aligned}$$

literally says

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$A \qquad x \quad = b$$

Properties If defined, then the following properties hold

1. $(AB)C = A(BC)$
2. $A(B+C) = AB + AC$
3. $(A+B)C = AC + BC$
4. $(\gamma A)B = \gamma(AB)$ where $\gamma \in \mathbb{R}$

Proof of #2 Let $D = A(B+C)$
Let $E = AB + AC$

Then $d_{ij} = \sum_{k=1}^{m} a_{ik}(b_{kj} + c_{kj}) = \sum_{k=1}^{m} [a_{ik}b_{kj} + a_{ik}c_{kj}]$

And $e_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj} + \sum_{k=1}^{m} a_{ik}c_{kj}$

And these are equal! $\square$

Proof of #1 Let A be an $m \times n$ matrix
" B " $n \times p$ "
" C " $p \times q$ "

Let $R = AB$, Let $S = (AB)C$, Let $U = BC$, $T = A(BC)$

Then $r_{ik} = \sum_{l=1}^{n} a_{il}b_{lk}$

So $s_{ij} = \sum_{k=1}^{p} \boxed{r_{ik}} c_{kj} = \sum_{k=1}^{p}\left(\sum_{l=1}^{n} a_{il}b_{lk}\right)c_{kj} = \sum_{k=1}^{p}\sum_{l=1}^{n} a_{il}b_{lk}c_{kj}$

And $u_{lj} = \sum_{k=1}^{p} b_{lk}c_{kj}$

So $t_{ij} = \sum_{l=1}^{n} a_{il}\boxed{u_{lj}} = \sum_{l=1}^{n} a_{il}\left(\sum_{k=1}^{p} b_{lk}c_{kj}\right)$

$= \sum_{l=1}^{n}\sum_{k=1}^{p} a_{il}b_{lk}c_{kj}$ $\square$

These properties are enough to prove the following fact about systems of linear eqns:

__Thm__ If $Ax = b$ is a system of linear equations, then it either has 0, 1 or ∞ solutions.

__proof__ We will show that if it has at least 2 solutions, then it has infinitely many solutions.

Thus suppose u and v are solutions and that $u \neq v$

In other words, $Au = b$ and $Av = b$

Then $A(u-v) = Au - Av = b - b = \mathbf{0}$

Then, if γ is <u>any</u> real number,

$$A\big(u + \gamma(u-v)\big) = Au + A\big(\gamma(u-v)\big)$$
$$= Au + \gamma A(u-v)$$
$$= Au + \gamma \mathbf{0}$$
$$= Au + \mathbf{0}$$
$$= Au$$
$$= b$$

Therefore, $u + \gamma(u-v)$ is a solution, and since $u - v \neq \mathbf{0}$, all of these are different $\square$

<u>Def</u> $Ax = \mathbf{0}$ is called a <u>homogeneous</u> system of linear equations.

<u>Theorem</u> A homogeneous system has either 1 or ∞ solutions.

<u>proof</u> First, $\mathbf{0}$ is always a solution.

Next, if u is a solution and $u \neq \mathbf{0}$,

then $Au = \mathbf{0}$.

So $A(\gamma u) = \gamma(Au) = \gamma \mathbf{0} = \mathbf{0}$

Thus γu is a solution for all $\gamma \in \mathbb{R}$ $\square$

2.3 Matrix Inverses

We will now discuss the (multiplicative) <u>inverse</u> of a matrix. This will allow us to give an even <u>more</u> robust analysis of SLE's.

From now on, all matrices are square unless otherwise specified.

<u>Defn</u> If $A = (a_{ij})$ is an $n \times n$ matrix, then the entries a_{ii} together make up the <u>diagonal</u> of A.

Defn I_n is the $n\times n$ matrix with 1 on each diagonal entry and 0 on every other entry.

[Literally $I_n = (a_{ij})$ where $a_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$]

Ex $I_1 = (1)$

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Prop If A is an $n\times n$ matrix, then $AI_n = I_n A = A$

Therefore, we call I_n the $n\times n$ (multiplicative) identity matrix.

check for 3×3

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Definition If A, B are $n\times n$ matrices an $AB = I_n = BA$, then we say A and B are inverses.

In this case, we write $B = A^{-1}$

Question Does A^{-1} always exist? NO

But if A^{-1} exists, we say that A is invertible.

Lemma If A^{-1} exists, then it is unique.

proof Suppose $A^{-1} = B$ and $A^{-1} = C$.

Then $B = BI = B(AC) = (BA)C = IC = C$ $\square$