Recall that if $AB = I = BA$, then A & B are inverses and we write $B = A^{-1}$

Thm ① If A^{-1} exists, then it is unique

pf last time $\square$

Thm ② If A^{-1} and B^{-1} both exist, then $(AB)^{-1} = B^{-1}A^{-1}$

pf $(AB)(B^{-1}A^{-1}) \overset{\text{Associativity}}{=} A \underbrace{BB^{-1}} A^{-1} = \underbrace{AI} A^{-1} = AA^{-1} = I$

AND $(B^{-1}A^{-1})(AB) = B^{-1} \underbrace{A^{-1}A} B = B^{-1} \underbrace{IB} = B^{-1}B = I$ $\square$

Corollary $(A^{-1})^{-1} = A$

proof $(A^{-1})^{-1}$ and A are both inverses of A^{-1}, so they must be equal, by Thm ①.

Finding A^{-1}

Our method will turn out to use elementary row operations!

Let's start with the 2×2 case:

If we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we want to find $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ so that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ae + bg \\ ce + dg \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{AND} \quad \begin{pmatrix} af + bh \\ cf + dh \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\times \left(\begin{array}{cc|c} ae & +bg & 1 \\ ce & +dg & 0 \end{array}\right) \qquad \text{AND} \qquad \times \left(\begin{array}{cc|c} af & +bh & 0 \\ cf & +dh & 1 \end{array}\right)$$

(variables)

$$\left(\begin{array}{cc|c} a & b & 1 \\ c & d & 0 \end{array}\right) \qquad \left(\begin{array}{cc|c} a & b & 0 \\ c & d & 1 \end{array}\right)$$

Since both of these have the same coefficients, they will be solved with the same sequence of row ops. Then at the end, we will have

$$\left(\begin{array}{cc|c} 1 & 0 & e \\ 0 & 1 & g \end{array}\right) \qquad \left(\begin{array}{cc|c} 1 & 0 & f \\ 0 & 1 & h \end{array}\right)$$

In other words, we will use row ops to get from

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \quad \text{to} \quad \left(\begin{array}{cc|cc} 1 & 0 & e & f \\ 0 & 1 & g & h \end{array}\right)$$

<u>Summary</u> To find A^{-1}:

1. Start with $(A|I)$
2. Row Reduce to $(I|B)$
3. Conclude that $B = A^{-1}$

Let's use this to find a formula for A^{-1} for 2×2 matrices.

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \quad \begin{array}{l} \frac{1}{a}R_1 \to R_1 \\ \frac{1}{c}R_2 \to R_2 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 1 & \frac{d}{c} & 0 & \frac{1}{c} \end{array}\right) \quad R_2 - R_1 \to R_2$$

$$\left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{ac} & -\frac{1}{a} & \frac{1}{c} \end{array}\right) \quad \frac{ac}{ad-bc} R_2 \to R_2$$

$$\left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) \quad R_1 - \frac{b}{a}R_2 \to R_1$$

$$\left(\begin{array}{cc|cc} 1 & 0 & \boxed{\frac{1}{a} + \frac{bc}{a(ad-bc)}} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

$$\frac{ad-bc}{a(ad-bc)} + \frac{bc}{a(ad-bc)} = \frac{\cancel{a}d}{\cancel{a}(ad-bc)}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

Thus, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Yay!

Check $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

$$= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example Find $\begin{pmatrix} 1 & 2 \\ -1 & 6 \end{pmatrix}^{-1}$

$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & 6 & 0 & 1 \end{array}\right)$ $R_2 + R_1 \to R_2$

$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 8 & 1 & 1 \end{array}\right)$ $\frac{1}{8}R_2 \to R_2$

$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{8} & \frac{1}{8} \end{array}\right)$ $R_1 - 2R_2 \to R_1$

$\left(\begin{array}{cc|cc} 1 & 0 & \frac{3}{4} & -\frac{1}{4} \\ 0 & 1 & \frac{1}{8} & \frac{1}{8} \end{array}\right)$

So $\begin{pmatrix} 1 & 2 \\ -1 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix}$

Find $\begin{pmatrix} 1 & 0 & 5 \\ 1 & 1 & 0 \\ 3 & 2 & 6 \end{pmatrix}^{-1}$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 6 & 0 & 0 & 1 \end{array}\right)$ $R_3 - 3R_2 \to R_3$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 6 & 0 & -3 & 1 \end{array}\right)$ $R_2 - R_1 \to R_2$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right)$ $R_1 - 5R_3 \to R_1$, $R_2 + 5R_3 \to R_2$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 10 & -5 \\ 0 & 1 & 0 & -6 & -9 & 5 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right)$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & 1 & 0 \\ 0 & -1 & 6 & 0 & -3 & 1 \end{array}\right) \quad R_3 + R_2 \to R_3$$

So $\begin{pmatrix} 1 & 0 & 5 \\ 1 & 1 & 0 \\ 3 & 2 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} 6 & 10 & -5 \\ -6 & -9 & 5 \\ -1 & -2 & 1 \end{pmatrix}$

$$\left(\begin{array}{cccc|cccc} 1 & 3 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 & 0 & 0 \\ -2 & -6 & 3 & 2 & 0 & 0 & 1 & 0 \\ 3 & 5 & 8 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_3 + 2R_1 \to R_3 \\ R_4 - 3R_1 \to R_4 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 3 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 2 & 0 & 1 & 0 \\ 0 & -4 & 8 & 0 & -3 & 0 & 0 & 1 \end{array}\right) \quad R_4 + 4R_2 \to R_4$$

$$\left(\begin{array}{cccc|cccc} 1 & 3 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 & -3 & 4 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - \frac{1}{4}R_4 \to R_1 \\ R_2 - \frac{1}{4}R_4 \to R_2 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & \frac{7}{4} & -1 & 0 & -\frac{1}{4} \\ 0 & 1 & -2 & 0 & \frac{3}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 3 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 & -3 & 4 & 0 & 1 \end{array}\right) \quad R_2 + \frac{2}{3}R_3 \to R_2$$

$$\left(\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & \frac{7}{4} & -1 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{25}{12} & 0 & \frac{2}{3} & -\frac{1}{4} \\ 0 & 0 & 3 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 & -3 & 4 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - 3R_2 \to R_1 \\ \\ \frac{1}{3}R_3 \\ -\frac{1}{4}R_4 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{18}{4} & -1 & -2 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{25}{12} & 0 & \frac{2}{3} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & \frac{3}{4} & -1 & 0 & -\frac{1}{4} \end{array}\right)$$

When is a matrix invertible? (This will also motivate Ch 3)

First, let's talk about elementary matrixes.

Defn An elementary matrix is obtained by performing one elementary row operation on I.

Ex

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 + 3R_2 \to R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} 4R_2 \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

why are these important?

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g+3d & h+3e & i+3f \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ 4d & 4e & 4f \\ g & h & i \end{pmatrix}$$

So each elem. row op. can be performed by multiplying on the left by the corresponding elem matrix.

Lemma Every elementary matrix is invertible

Example

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Definition A and B are Row Equivalent if there is a sequence of elementary matrices $E_1, \ldots, E_k$ so that $A = (E_k \cdots E_1) B$

(and therefore $E_k^{-1} A = E_{k-1} \cdots E_1 B$

$\vdots$

$(E_1^{-1} E_2^{-1} \cdots E_k^{-1}) A = B$)