$$(AB)^{-1} = B^{-1}A^{-1}$$

$$Ax = b$$

Probably a square matrix; column matrices

Theorem Let A be a matrix. Then the following are equivalent

(a) A is invertible

(b) $Ax = 0$ has only the trivial solution

(c) A and I are row equivalent

(d) For any b, $Ax = b$ has a solution

(e) For any b, $Ax = b$ has the unique solution $x = A^{-1}b$

Proof $(a \Rightarrow b)$

Assume A has an inverse. Suppose x_0 is a solution to $Ax = 0$

Then

$$x_0 = Ix_0 = (A^{-1}A)x_0 = A^{-1}(Ax_0) = A^{-1}0 = 0$$

Tom's Lemma

If $A^2 = 0$, then A is not invertible

pf Suppose not.

$$A \cdot A = 0$$

$$\Rightarrow A^{-1}AA = A^{-1}0$$

$$\Rightarrow IA = 0$$

$$\Rightarrow A = 0 \quad \square$$

$(b \Rightarrow c)$ Suppose $Ax = 0$ has only the trivial solution. Then $Ax = 0$ can be row reduced to the set of equations $\{x_1 = 0, x_2 = 0, \dots, x_n = 0\}$

In other words

$$(A \mid 0) \xrightarrow[\text{Reducable}]{\text{Row}} (I \mid 0)$$

Hence $A \sim I$.

$(c \Rightarrow a)$ Suppose $A \sim I$

Then $A = E_k \cdots E_2 E_1 I$

Since elementary matrices are invertible $\Rightarrow A^{-1} = E_1^{-1}E_2^{-1} \cdots E_k^{-1}$

$(a \Rightarrow e)$ Existence

Let $x_0 = A^{-1}b$

Then $Ax_0 = AA^{-1}b = Ib = b$

Uniqueness

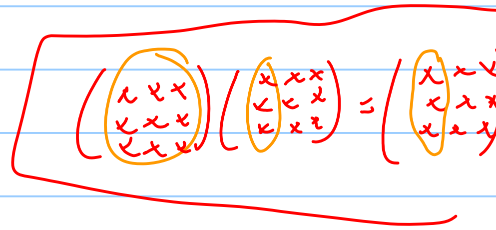
Suppose y is a solution.

Then $Ay = b$

So $A^{-1}Ay = A^{-1}b$

$\Rightarrow Iy = A^{-1}b$

$\Rightarrow y = A^{-1}b$



Therefore $y = x_0$

$(e \Rightarrow d)$ Immediate.

$(d \Rightarrow a)$ Suppose that for any b, $Ax = b$ has a solution.

Let $b_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← ith position

Then $Ax = b_i$ has a solution.

So let u_i be a solution to $Ax = b_i$

Then $A(u_1 \ u_2 \ \cdots \ u_n) = (b_1 \ b_2 \ \cdots \ b_n) = I$

Therefore $(u_1 \ u_2 \cdots u_n) = A^{-1}$ $\square$

Lemma If $BA = I$ and $AC = I$ then $B = C$.

proof $B = BI = B(AC) = (BA)C = IC = C$

Lemma If $BA = I$ then $AB = I$

pf Delayed.

Applications

Chapter 1 # 18 Let x_1 be the price of bannnas

" x_2 " oranges

" x_3 " apples

uses = produces

A: $0.2x_1 + 0.5x_2 + 0.2x_3 = x_1$

B: $0.3x_1 + 0.3x_2 + 0.4x_3 = x_2$

C: $0.5x_1 + 0.2x_2 + 0.4x_3 = x_3$

$$Ax = x$$

$$Ax - x = 0$$

$$Ax - Ix = 0$$

$$(A-I)x = 0$$

Since this should have ∞-many solutions, $(A-I)$ should be non-invertible.

(challenge problem If columns of A all sum to 1, then $(A-I)^{-1}$ DNE)

That was a closed Leontief model.

An open Leontief model includes outside demand.

Chapter 2; Exercise 10

Let x_1 be the value of the output of industry A

" x_2 " B

" x_3 " C

$$x_1 = 0.65x_2 + 0.55x_3 + 500000$$

$$x_2 = 0.25x_1 + 0.05x_2 + 0.10x_3 + 250000$$

$$x_3 = 0.25x_1 + 0.05x_2$$

$$x = Ax + d$$

$$Ix - Ax = d$$

$$(I-A)x = d$$

so $x = (I-A)^{-1}d$

(HW #9a as stated, but also with $d = \begin{pmatrix} 60\,000 \\ 30\,000 \end{pmatrix}$ and $d = \begin{pmatrix} 50\,000 \\ 40\,000 \end{pmatrix}$)

Matrix transformations of the plane

Let's make some elementary geometric observations of the effects of multiplication by 2×2 matrices on points in the plane. For this section, Assume A is a 2×2 matrix

First, notice that

$\rightarrow$ $A\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} = A\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + A\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$\rightarrow$ $A\begin{pmatrix} cx_1 \\ cy_1 \end{pmatrix} = cA\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

So this operation is linear.

Let's see some examples

$\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\-y\end{pmatrix}$ Reflection across x-axis

$\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-x\\y\end{pmatrix}$ Reflection across y-axis

$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-x\\-y\end{pmatrix}$ Reflection across origin

$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}y\\x\end{pmatrix}$ Reflection across $x=y$

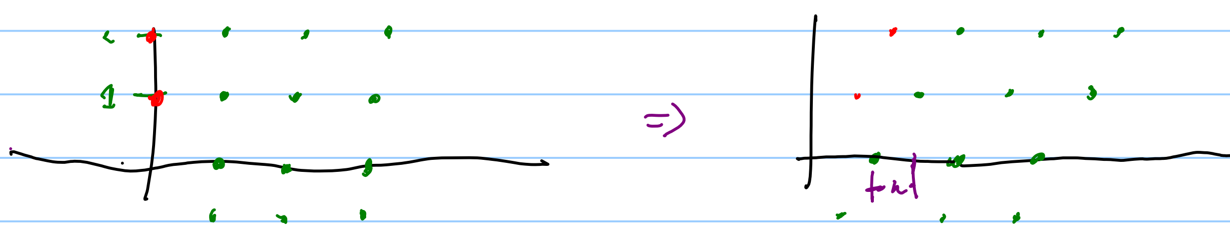
$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$ Identity

$\begin{pmatrix}k&0\\0&k\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}kx\\ky\end{pmatrix}$ Dilation by k

Similarly for y

$\begin{pmatrix}k&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}kx\\y\end{pmatrix}$ Dilation on x coordinate

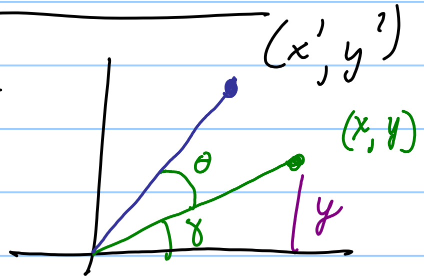
$\begin{pmatrix}1&k\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+ky\\y\end{pmatrix}$??



The name of this is shear.

Rotations?

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$



$\sin\gamma=\frac{y}{\sqrt{x^2+y^2}}$ $\cos\gamma=\frac{x}{\sqrt{x^2+y^2}}$ $\tan\gamma=\frac{y}{x}$

$\sin(\gamma+\theta)=\frac{y'}{\sqrt{x^2+y^2}}$ | $\cos(\gamma+\theta)=\frac{x'}{\sqrt{x^2+y^2}}$

$\sin\gamma\cos\theta+\cos\gamma\sin\theta=\frac{y'}{\sqrt{x^2+y^2}}$ | $\cos\gamma\cos\theta-\sin\gamma\sin\theta=$

$$\frac{y}{\sqrt{x^2+y^2}}\cos\theta + \frac{x}{\sqrt{x^2+y^2}}\sin\theta = \frac{y'}{\sqrt{x^2+y^2}} \quad \Bigg| \quad \frac{x}{\sqrt{x^2+y^2}}\cos\theta - \frac{y}{\sqrt{x^2+y^2}}\sin\theta = \frac{x'}{\sqrt{x^2+y^2}}$$

$$y' = x\sin\theta + y\cos\theta \quad \Big| \quad x' = x\cos\theta - y\sin\theta$$

Thus $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

So $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$
