76 $Ax=b$

$x=A^{-1}b$

Last Time we proved that if $BA = I = AC$, then $B=C$.

Lemma If A has a right inverse, then it has a left inverse.

Proof Suppose $AC=I$

Suppose toward a contradiction that A has no left inverse.
Thus when we row reduce A, we get an all zero row.
So there is a sequence of Elem. Mat. $E_1 \dots E_k$
so that $E_k \dots E_1 A$ has an all zero row.
So $E_k \dots E_1 A C$ has the same all all zero row
But $E_k \dots E_1 A C = E_k \dots E_1 I = E_k \dots E_1$ is invertible
contradiction!
A matrix with an all zero row is not invertible.

$\square$

Lemma If A has a left inverse, then it has a right inverse
~~of~~ Challenge Problem.

Matrix Transformations of the plane (cont.)

Since $A\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ for any A, we cannot do translations this way.
So how can we incorporate them?
We will replace each $\begin{pmatrix}x\\y\end{pmatrix}$ by $\begin{pmatrix}x\\y\\1\end{pmatrix}$

and each transformation $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ by $\begin{pmatrix}a&b&0\\c&d&0\\0&0&1\end{pmatrix}$

Then $\begin{pmatrix}a&b&0\\c&d&0\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\1\end{pmatrix} = \begin{pmatrix}ax+by\\cx+dy\\1\end{pmatrix}$

But now if we want to translate $\rightarrow h,\ \uparrow k$

we use $\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+h \\ y+k \\ 1 \end{pmatrix}$

(See 2.10 for nice diagrams)

$$\begin{pmatrix} a & b & h \\ c & d & k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \cancel{\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & h \\ 0 & 0 & 1 \end{pmatrix} = a \quad b \quad ah+bk}$$

$$= \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

LU-factorization

Suppose we want to solve $Ax = b$ for various b.

<u>Ans</u> Find A^{-1} to get $A^{-1}b$!!

But what if A is not invertible? Or what if A is not square??

<u>Example</u> $Ax = b$, where

$$A = \begin{pmatrix} 3 & -1 & 2 & -4 & 1 \\ -3 & 3 & -5 & 5 & -2 \\ 6 & -4 & 11 & -10 & 6 \\ -6 & 8 & -21 & 13 & -9 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ -2 \\ 9 \\ -15 \end{pmatrix}$$

How many entries does x have? 5

Suppose a genie tells you that $A = LU$, where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -2 & 3 & -2 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 3 & -1 & 2 & -4 & 1 \\ 0 & 2 & -3 & 1 & -1 \\ 0 & 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Then $L\underbrace{Ux}=b$

So (1) Solve for Ux first? Let $Ux = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -2 & 3 & -2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 9 \\ -15 \end{pmatrix}$$

Front solve!

Row 1 $\quad y_1 = 1$

Row 2 $\quad -y_1 + y_2 = -2$

$-1 + y_2 = -2$

$y_2 = -1$

Row 3 $\quad 2 + 1 + y_3 = 9$

$y_3 = 6$

Row 4 $\quad -2 - 3 - 12 + y_4 = -15$

$y_4 = 2$

$\Rightarrow \quad Ux = \begin{pmatrix} 1 \\ -1 \\ 6 \\ 2 \end{pmatrix}$

Step (2): Solve for x: $\begin{pmatrix} 3 & -1 & 2 & -4 & 1 \\ 0 & 2 & -3 & 1 & -1 \\ 0 & 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 6 \\ 2 \end{pmatrix}$

Now Backsolve

Row 4 $\quad 2x_5 = 2$

$x_5 = 1$

Let $x_4 = S$

Row 3 $\quad 4x_3 - S + 3 = 6$

$x_3 = \frac{S}{4} + \frac{3}{4}$

Row 2 $\quad 2x_2 - \frac{3S}{4} - \frac{9}{4} + S - 1 = -1$

$2x_2 = -\frac{S}{4} + \frac{9}{4}$

$x_2 = \frac{-S}{8} + \frac{9}{8}$

etc.

So, this is not quite as fast as multiplying $A^{-1}b$, but it's faster than solving $Ax=b$ from scratch.

What about the genie?
How can we factor $A = LU$, where
L is lower unit triangular
and U is upper triangular?

Ans **LU Factorization Algorithm**

(QREF : Pivot entries need not be 1)

1. Reduce A to quasi row echelon form only using the row operation "add a multiple of a higher row to a lower row." This will give us U.

2. What will L be? If $U = E_k \cdots E_1 A$
Then $A = \underbrace{(E_k \cdots E_1)^{-1}}_{L} U$

So to find L, we start with I and record all of the row operations used.

Our Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 2 & -4 & 1 \\ -3 & 3 & -5 & 5 & -2 \\ 6 & -4 & 11 & -10 & 6 \\ -6 & 8 & -21 & 13 & -9 \end{pmatrix} \quad R_2 + R_1 \to R_2$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{-1} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 2 & -4 & 1 \\ 0 & 2 & -3 & 1 & -1 \\ 6 & -4 & 11 & -10 & 6 \\ -6 & 8 & -21 & 13 & -9 \end{pmatrix} \quad R_4 + 2R_1 \to R_4$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 2 & -4 & 1 \\ 0 & 2 & -3 & 1 & -1 \\ 6 & -4 & 11 & -10 & 6 \\ 0 & 6 & -17 & 5 & -7 \end{pmatrix} \quad R_3 - 2R_1 \to R_3$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 2 & -4 & 1 \\ 0 & 2 & -3 & 1 & -1 \\ 0 & -2 & 7 & -2 & 4 \\ 0 & 6 & -17 & 5 & -7 \end{pmatrix} \quad \begin{matrix} R_3 + R_2 \to R_3 \\ R_4 - 3R_2 \to R_4 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 2 & -4 & 1 \\ 0 & 2 & -3 & 1 & -1 \\ 0 & 0 & 4 & -1 & 3 \\ 0 & 0 & -8 & 2 & -4 \end{pmatrix} \quad R_4 + 2R_3 \to R_4$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -2 & 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 2 & -4 & 1 \\ 0 & 2 & -3 & 1 & -1 \\ 0 & 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

L U Ta da!

Example 2

$$A = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 4 & 3 & 5 & 4 \\ 4 & 3 & 5 & 7 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 9 \\ 18 \end{pmatrix}$$

Find all solutions!

Factor A

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 & 1 \\ 4 & 3 & 5 & 4 \\ 4 & 3 & 5 & 7 \end{pmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 5 \end{pmatrix} \quad R_3 - R_2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Solve for $\underbrace{Ux}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \\ 18 \end{pmatrix}$$

Front Solve

Row 1 $y_1 = 1$

Row 2 $2 + y_2 = 9$

$y_2 = 7$

Row 3 $2 + 7 + y_3 = 18$

$y_3 = 9$

$$Ux = \begin{pmatrix} 1 \\ 7 \\ 9 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 9 \end{pmatrix}$$

Back solve

Row 3 $3x_4 = 9$

$x_4 = 3$

Let $x_3 = s$

Row 2 $x_2 + s + 6 = 7$

$x_2 = 1 - s$

Row 1 $2x_1 + 1 - s + 2s + 3 = 1$

$2x_1 = -s - 3$

$x_1 = \frac{-s-3}{2}$

General solution $(x_1, x_2, x_3, x_4) = \left(\frac{-s-3}{2}, 1-s, s, 3\right)$

Chapter 3 Determinants

Inverse of 1×1 matrix $(a)^{-1} = \left(\frac{1}{a}\right)$ Exists iff $a \neq 0$

Inverse of 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ Exists iff $ad - bc \neq 0$

– We will define determinants recursively in terms of cofactors

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

checker-board pattern for +/- :

$$\begin{pmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + & \cdots \\ - & + & - & + & - \\ + & - & + & - & + \\ & & \vdots \end{pmatrix}$$

Example $\det\begin{pmatrix} 1 & 4 & 6 \\ 2 & 3 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$$= 0 - (-1)\det\begin{pmatrix} 1 & 6 \\ 2 & 0 \end{pmatrix} + 1\det\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

$$= (1\cdot 0 - 6\cdot 2) + (1\cdot 3 - 4\cdot 2)$$

$$= -17$$