$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Determinants

Let A be an $n \times n$ matrix, Then we write A_{ij} for the $(n-1)\times(n-1)$ matrix obtained by deleting row i & column j.

Now let $1 \le i \le n$. We define the determinant of A recursively as follows:

$$\text{(*)} \quad \det(A) = \sum_{j=1}^{n} (-1)^{(i+j)} a_{ij} \det(A_{ij}) \qquad \text{if } n > 1$$

$$\text{(*)} \quad \det(A) = a_{11} \qquad \text{if } n = 1$$

A_{ij} is called a minor of A

$C_{ij} = (-1)^{i+j} \det(A_{ij})$ is called a cofactor of A

Theorem $\det(A)$ is the same no matter which row (or column) you define it with.

proof Delayed

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Example Find $\det \begin{pmatrix} 1 & 4 & 6 \\ 2 & 3 & 0 \\ 0 & -1 & 1 \end{pmatrix} = -17$

Top row
$$\det\begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix} - 4 \det\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + 6 \det\begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}$$
$$= 3 - 4(2) + 6(-2)$$
$$= -17$$

Column 3
$$6 \det\begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} - 0 \det\begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix} + 1 \det\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$
$$= 6(-2) - 0 + (-5)$$
$$= -17$$

Example Find $\det \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 4 & 0 & 6 \\ 9 & 3 & 8 & 26 \\ 8 & -11 & 0 & -1 \end{pmatrix}$

$$= 8 \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 8 & -11 & -1 \end{pmatrix}$$

$$= 8 \left[\det \begin{pmatrix} 4 & 6 \\ -11 & -1 \end{pmatrix} - 0 + 8 \det \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \right]$$

$$= 8 \left[62 + 8(0) \right]$$

$$= 496$$

<u>Lesson</u> Strategize!

<u>Target Theorem ①</u> If A is a square matrix, then A^{-1} exists $\Leftrightarrow$ $\det(A) \neq 0$

<u>Target Theorem ②</u> $\det(AB) = \det(A) \cdot \det(B)$

<u>Lemma</u> Suppose the square matrix A has a row or a column that is all zeros. Then $\det(A) = 0$.

<u>Proof</u> Calculate the determinate using that row/column $\square$

$\begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$ <u>Defn</u> A is <u>upper triangular</u> if $a_{ij} = 0$ for all $i > j$

$\begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix}$ <u>Defn</u> A is <u>lower triangular</u> if $a_{ij} = 0$ for all $j > i$

<u>Lemma</u> Suppose A is either upper triangular or lower triangular. Then $\det(A)$ is the product of the diagonal entries.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 6 & 0 & 0 \\ 7 & 3 & 2 & 0 \\ 10 & 11 & 12 & 13 \end{pmatrix}$$

<u>Proof</u> $\det(A) = a_{11} \cdot \det \begin{pmatrix} a_{22} & & 0 \\ & \ddots & \\ X & & a_{nn} \end{pmatrix}$

$$= a_{11} \cdot a_{22} \cdot \det \begin{pmatrix} a_{33} & & 0 \\ & \ddots & \\ X & & a_{nn} \end{pmatrix}$$

$$= \dots = a_{11} a_{22} \cdots a_{nn} \quad \square$$

The key fact will be what elementary matrices do to the determinant!

<u>Thm</u> Suppose A is a square matrix:

① If B is obtained by switching 2 rows of A, then $\det(B) = -\det(A)$

② If B is obtained by adding a nonzero multiple of one row of A to another, then $\det(B) = \det(A)$

③ If B is obtained from A by multiplying a row by the constant k, then $\det(B) = k \cdot \det(A)$

proof ① Induction on n. Base Case $\det\begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - ad = -\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Inductive Step Suppose $n \geq 3$. Suppose we switch rows i and j. Let k be any other row. Expand along row k:

$$\det(B) = \sum_{l=1}^{n} b_{kl} (-1)^{l+k} \det(B_{kl})$$

$$= \sum_{l=1}^{n} a_{kl} (-1)^{l+k} \det(B_{kl}) \quad \text{since row } k \text{ is unchanged}$$

$$= \sum_{l=1}^{n} a_{kl} (-1)^{l+k} \left(-\det(A_{kl})\right) \quad \text{by induction}$$

$$= -\sum_{l=1}^{n} a_{kl} (-1)^{l+k} \det(A_{kl}) = -\det(A)$$

$R_1 + 5R_2 \to R_1$

~~$5R_2 + R_1 \to R_2$~~

② Induction on n. Base Case $\det\begin{pmatrix} a+hc & b+hd \\ c & d \end{pmatrix}$

$$= (a+hc)d - (b+hd)c$$

$$= ad + hcd - bc - hcd = ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Inductive Step $(n \geq 3)$ Suppose B is obtained from A by operation $R_i + hR_j \to R_i$

Let k be any row besides i and j.

$$\det(B) = \sum_{l=1}^{n} b_{kl} (-1)^{l+k} \det(B_{kl})$$

$$= \sum_{l=1}^{n} a_{kl} (-1)^{l+k} \det(A_{kl}) \quad \begin{array}{l} a_{kl} = b_{kl} \text{ since row } k \text{ unchanged} \\ \det(B_{kl}) = \det(A_{kl}) \text{ by induction} \end{array}$$

$$= \det(A)$$

③ Suppose B is obtained from A by operation $kR_i \to R_i$

Expand along row i:

$$\det(B) = \sum_{j=1}^{n} b_{ij} (-1)^{i+j} \det(B_{ij}) = \sum_{j=1}^{n} k a_{ij} (-1)^{i+j} \det(A_{ij}) = k\det(A)$$

other rows are unchanged

$\square$

Thm Similar for colums ~~proof~~ similar.

Example
$$\det\begin{pmatrix} 2 & 1 & 5 & 1 & 3 \\ 2 & 1 & 5 & 1 & 2 \\ 4 & 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 0 & 6 \\ 2 & 1 & 6 & \pi & 7 \end{pmatrix} \xrightarrow{R_1 - R_2 \to R_1} = \det\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 5 & 1 & 2 \\ 4 & 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 0 & 6 \\ 2 & 1 & 6 & \pi & 7 \end{pmatrix}$$

$$= \det\begin{pmatrix} 2 & 1 & 5 & 1 \\ 4 & 3 & 2 & 1 \\ 4 & 3 & 2 & 0 \\ 2 & 1 & 6 & \pi \end{pmatrix} \xrightarrow{R_2 - R_3 \to R_2} = \det\begin{pmatrix} 2 & 1 & 5 & 1 \\ 0 & 0 & 0 & 1 \\ 4 & 3 & 2 & 0 \\ 2 & 1 & 6 & \pi \end{pmatrix}$$

$$= \det\begin{pmatrix} 2 & 1 & 5 \\ 4 & 3 & 2 \\ 2 & 1 & 6 \end{pmatrix} \xrightarrow{R_3 - R_1 \to R_3} = \det\begin{pmatrix} 2 & 1 & 5 \\ 4 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = 2$$

Defn If $A = (a_{ij})$ then the transpose of A is defined as $A^t = (a_{ji})$

Thm $\det A = \det A^t$

proof Expanding along row i of A is the same as expanding along column i of A^t □

Lemma 1 If E is an elementary matrix, then

(A) $\det(E) = 1$ if E corresponds to $R_i + kR_j \to R_i$

(B) $\det(E) = -1$ if E corresponds to $R_i \leftrightarrow R_j$

(C) $\det(E) = k$ if E corresponds to $kR_i \to R_i$

Sketch Proof (A) **Examples**

$$E = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{then } \det E = 1$$

(B)
$$\det\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$$

(C)
$$\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 5$$

□

Lemma 2 If E is elementary then $\det(EA) = \det(E)\cdot\det(A)$

$\square$

Proof of target 2 $\left(\det(AB) = \det(A)\det(B)\right)$

Let A' be A in RREF.

Suppose $A = E_1 \dots E_k A'$ where the E's are elementary

Case 1 $\det(A) = 0$

$\Rightarrow \det(A') = 0$.

Then since A' is in RREF, it has an all zero row.

$\Rightarrow$ $A'B$ has an all zero row $\Rightarrow \det(A'B) = 0$

$\Rightarrow$ $\det(AB) = \det(E_1 \dots E_k A'B)$

$= \det(E_1) \dots \det(E_k)\det(A'B) = 0$

Case 2 $\det(A) \neq 0$

$\Rightarrow \det(A') \neq 0$

$\Rightarrow A' = I$

$\Rightarrow AB = E_1 \dots E_k B$

$\Rightarrow \det(AB) = \det(E_1) \dots \det(E_k)\det(B)$

$= \det(E_1 \dots E_k)\det(B)$

$= \det(A)\det(B)$ $\square$