$(AB)^{-1} = B^{-1}A^{-1}$

Last Time we proved $\det(AB) = \det(A)\det(B)$.

Now we will show:

Theorem A^{-1} exists iff $\det(A) \neq 0$

proof ($\Leftarrow$) Suppose $\det(A) \neq 0$.

Suppose $B = E_k \dots E_1 A$ is in RREF

Then $\det(B) = \underbrace{\det(E_k) \dots \det(E_1)\det(A)}$

Since all of these are not zero, then $\det(B) \neq 0$. Thus B has no all-zero rows, and so $B = I$. Therefore

$$A^{-1} = (E_k \dots E_1)^{-1} I = E_1^{-1} E_2^{-1} \dots E_k^{-1}$$

In fact, this is how we usually find the inverse.

($\Rightarrow$) Suppose A^{-1} exists. Then $A \cdot A^{-1} = I$

so $\det(A A^{-1}) = \det(I)$

$\det(A)\det(A^{-1}) = 1$

Thus $\det(A) \neq 0$ $\square$

Summary TFAE:

a) A is invertible
b) $Ax = 0$ has only the trivial solution
c) $Ax = b$ has the unique solution $x = A^{-1}b$ (for every b)
d) $A \sim I$
e) $\det(A) \neq 0$

Ch 2 #15a

$$A_1^* = \begin{pmatrix} 1 & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & k_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{compression in } x_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ k_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{shear in } x_2}$$

$SR_i + R_j \to R_i$ is not allowed

Example of LU-factorization

$$A = \begin{pmatrix} 3 & 1 & 3 \\ 9 & 4 & 10 \\ 6 & -1 & 5 \end{pmatrix} \qquad b = \begin{pmatrix} 5 \\ 18 \\ 9 \end{pmatrix}$$

Step 1: factor A

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 3 \\ 9 & 4 & 10 \\ 6 & -1 & 5 \end{pmatrix}$$

$R_2 - 3R_1 \to R_2$

$R_3 - 2R_1 \to R_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & -3 & -1 \end{pmatrix}$$

$R_3 + 3R_2 \to R_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

L U

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}$$

Step 2: solve for Ux

$Ax = b$

$L(Ux) = b$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix} (Ux) = \begin{pmatrix} 5 \\ 18 \\ 9 \end{pmatrix}$$

Front Solve

Row 1: $y_1 = 5$

Row 2: $3(5) + y_2 = 18$

$y_2 = 3$

Row 3: $2 \cdot (5) - 3(3) + y_3 = 9$

$y_3 = 8$

Step 3 solve for x

$$Ux = \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix}$$

Back solve

Row 3: $2x_3 = 8$

$x_3 = 4$ etc.

$$\underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} 7 \\ 8 \end{pmatrix}}$$

$x_1 + 2x_2 + 3x_3 = 7$

$4x_1 + 5x_2 + 6x_3 = 8$

Back to our Regularly Scheduled ...

Adjoints!

We can use determinants to find A^{-1}.

Let $A = (a_{ij})$ be an $n \times n$ matrix.

Define $c_{ij} = (-1)^{i+j} \det(A_{ij})$

$=$ the cofactor of the entry a_{ij}

Then define the adjoint of A as

$$\operatorname{adj}(A) = (c_{ji}) = (c_{ij})^t$$

Then $A^{-1} = \left(\frac{1}{\det(A)}\right) \operatorname{adj}(A)$

Example Find $\begin{pmatrix} 1 & 0 & 5 \\ 1 & 1 & 0 \\ 3 & 2 & 6 \end{pmatrix}^{-1}$ using the adjoint.

First find cofactors: cofactor matrix $\begin{pmatrix} 6 & -6 & -1 \\ 10 & -9 & -2 \\ -5 & 5 & 1 \end{pmatrix}$

$$\operatorname{adj}(A) = \begin{pmatrix} 6 & 10 & -5 \\ -6 & -9 & 5 \\ -1 & -2 & 1 \end{pmatrix}$$

$\det(A) = 1$

Thus $\begin{pmatrix} 1 & 0 & 5 \\ 1 & 1 & 0 \\ 3 & 2 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} 6 & 10 & -5 \\ -6 & -9 & 5 \\ -1 & -2 & 1 \end{pmatrix}$ (Check!)

CRAMER's Rule

Suppose we want to solve $Ax = b$.

Write $A_j(b)$ to be the matrix A with the j^{th} column replaced by B.

Then $x_i = \dfrac{\det(A_i(b))}{\det(A)}$

Example

$$x_1 - x_2 = 1$$
$$x_2 + 2x_3 = 2$$
$$2x_1 + 3x_3 = 3$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 2 & 0 & 3 & 3 \end{array}\right)$$

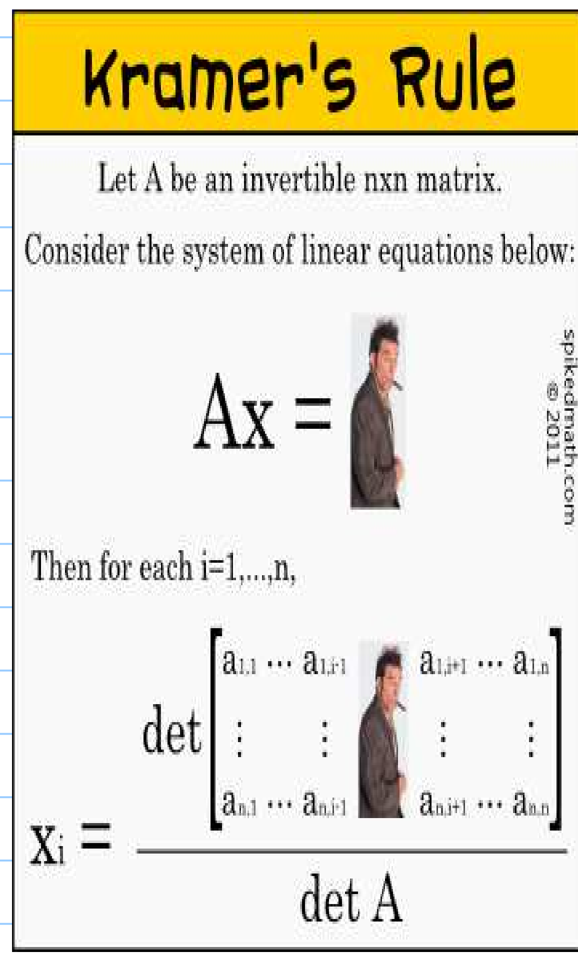
$$\det(A) = (1)(3) + (1)(-4) = -1$$

$$\det(A_1(b)) = \det \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 2 \\ 3 & 0 & 3 \end{pmatrix} = 3 + 0 = 3$$

$$\det A_2(b) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 2 & 3 & 3 \end{pmatrix} = (1)(0) - (1)(-4) = 4$$

$$\det(A_3(b)) = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 3 \end{pmatrix} = (1)(3) + (2)(-3) = -3$$

So Cramer's Rule says
$$x_1 = \frac{3}{-1} = -3$$
$$x_2 = \frac{4}{-1} = -4$$
$$x_3 = \frac{-3}{-1} = 3$$



Exercise 5B

Let a, e, k, m, n, r, s be nonzero constants and let

$$A = \begin{bmatrix} k/r & 2k/s & 0 & 0 & -k/r & -k/n \\ -1 & 0 & 0 & 0 & 1 & r/n \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2e/s & 0 & 0 & -e/n \\ 0 & 0 & 0 & 0 & 0 & r/n \end{bmatrix}.$$

Using Kramer's Rule, solve the linear system

$$Ax = K,$$

where $K = [k \quad r \quad a \quad m \quad e \quad r]^T$.

spikedmath.com
© 2011

<u>Theorem</u> If A^{-1} exists, then $A^{-1} = \left(\frac{1}{\det(A)}\right) \operatorname{adj}(A)$

<u>Proof</u> We want to show that $A\left(\left(\frac{1}{\det(A)}\right) \operatorname{adj}(A)\right) = I$

OR $\underbrace{A \operatorname{adj}(A)} = \det(A)\, I$

looking at LHS: $(a_{ij})(c_{ij}) := (b_{ij})$

where $b_{ij} = a_{i1} c_{1j} + a_{i2} c_{2j} + \dots + a_{in} c_{nj}$

<u>Case I $i = j$</u> Then $b_{ii} = \det(A)$

<u>Case II $i \neq j$</u> Then b_{ij} is the determinant of the matrix obtained by replacing row j with row i (calculated then at row j).

$$\begin{array}{c} i \\ j \\ \\ \end{array} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Since this matrix would then have 2 identical

Rows, its determinant is zero.

Therefore LHS = $\det(A)\, I$ $\square$