Recall $A^{-1} = \left(\frac{1}{\det(A)}\right) \operatorname{adj}(A)$ (proved last time)

Now, we will prove Cramer's Rule, namely, if $Ax = b$,
then $x_i = \dfrac{\det(A_i(b))}{\det(A)}$ where $A_i(b)$ is A, except column i is replaced by b.

__Proof__ Suppose $Ax = b$.
Then $x = A^{-1}b = \left(\frac{1}{\det(A)}\right) \operatorname{adj}(A)\, b$

Or
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$
where C_{ij} is the cofactor for a_{ij}

$$= \frac{1}{\det(A)} \begin{pmatrix} b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1} \\ b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2} \\ \vdots \\ b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn} \end{pmatrix}$$

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Thus $x_j = \frac{1}{\det(A)} \underbrace{\left(b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}\right)}_{*}$

To prove Cramer's Rule, we want $*$ to be $\det(A_j(b))$

So $\det(A_j(b)) = \sum_{i=1}^{n} b_i \underbrace{(-1)^{i+j} \det(A_{ij})}$

Along column j

$= \sum_{i=1}^{n} b_i \, C_{ij}$

$= *$

$\square$

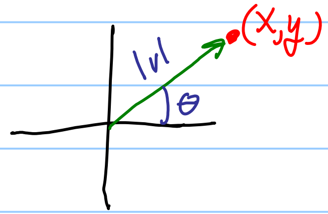
Chapter 4 Vectors

Informally a vector is a magnitude and a direction

Formally, we will consider vectors to be $n \times 1$ matrices.

However, to build our intuition, we will work in $\mathbb{R}^2$ for awhile

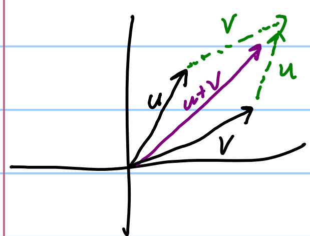
Vectors in $\mathbb{R}^2$



The vector v.

$$x = |v| \cos\theta$$
$$y = |v| \sin\theta$$

So we see that the vector $|v|, \theta$ can be thought of as the ordered pair (x,y) (More generally, ordered n-tuples)

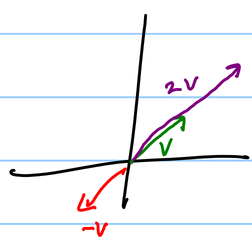


Adding vectors: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

Basic Properties If u, v, w are vectors, then

1. $(u+v)+w = u+(v+w)$
2. $u + 0 = u$ where 0 means $(0,0)$
3. for every $u \in \mathbb{R}^2$, there is a unique $v \in \mathbb{R}^2$ so that $u+v=0$
4. $u+v = v+u$

As with matrices, we also have scalar multiplication!



$$c(x,y) = (cx, cy)$$

Properties let $a,b,c \in \mathbb{R}$, let $u, v \in \mathbb{R}^2$

1. $c(u+v) = cu + cv$
2. $(a+b)u = au + bu$
3. $(ab)u = a(bu)$
4. $1u = u$

In $\mathbb{R}^2$, what is the magnitude of the vector $v = (x,y)$?

Definition $\|v\| = \sqrt{x^2+y^2}$ is the magnitude of v.

Definition Distance between $u = (u_1, u_2)$ and $v = (v_1, v_2)$ is

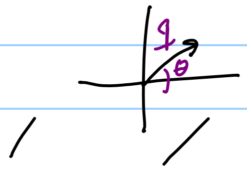
$$d(u,v) = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2}$$

Lemma $|c|\cdot\|v\| = \|cv\|$

pf $|c|\cdot\|v\| = |c|\sqrt{v_1^2+v_2^2} = \sqrt{c^2(v_1^2+v_2^2)} = \sqrt{(cv_1)^2+(cv_2)^2} = \|cv\| \quad \square$

Defn If $\|u\|=1$ we call u a unit vector

Example Unit vector in direction θ is $(\cos\theta, \sin\theta)$



Products of vectors?

Dot Product / Scalar Product

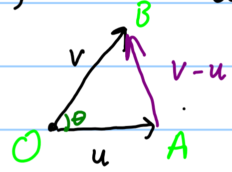
Definition Let $u, v \in \mathbb{R}^2$ be vectors.

Define $u \cdot v = u_1 v_1 + u_2 v_2$

More generally, if $u, v \in \mathbb{R}^n$ are vectors, define $u \cdot v = \sum_{i=1}^{n} u_i v_i$

Proposition If u, v are vectors in $\mathbb{R}^2$, and θ is the angle between them, then $u \cdot v = \|u\|\cdot\|v\|\cdot\cos\theta$

pf



By the Law of Cosines $\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 - 2\,\overline{OA}\,\overline{OB}\cos\theta$

$$\|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\cdot\|v\|\cdot\cos\theta$$

Thus $\|u\|\cdot\|v\|\cos\theta = \frac{1}{2}\left(\|u\|^2 + \|v\|^2 - \|v-u\|^2\right)$

$$= \frac{1}{2}\left(u_1^2+u_2^2 + v_1^2+v_2^2 - (v_1-u_1)^2 - (v_2-u_2)^2\right)$$

$$= \frac{1}{2}\left(u_1^2+u_2^2+v_1^2+v_2^2 - v_1^2 + \underbrace{2v_1u_1} - u_1^2 - v_2^2 + \underbrace{2v_2u_2} - u_2^2\right)$$

$$= \frac{1}{2}\left(2v_1u_1 + 2v_2u_2\right)$$

$$= v_1u_1 + v_2u_2$$

$$= u \cdot v \quad \square$$

ORTHO

In $\mathbb{R}^2$, we say that two vectors are orthogonal (right-angled) if $\theta = \frac{\pi}{2}$

Thus in $\mathbb{R}^2$, u, v are orthogonal iff $u \cdot v = 0$

o in general, we define u and v to be orthogonal if $u \cdot v = 0$

Proposition Suppose $u, v, w \in \mathbb{R}^2$ and $c \in \mathbb{R}$

(a) $u \cdot v = v \cdot u$

(b) $u \cdot (v + w) = u \cdot v + u \cdot w$

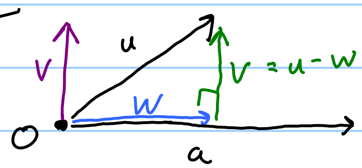
(c) $c\,(u \cdot v) = (cu) \cdot v = u \cdot (cv)$

(d) $u \cdot u = \|u\|^2 \geq 0$

(e) $u \cdot u = 0$ iff $u = 0$

proof of b
$$\begin{aligned} u \cdot (v+w) &= (u_1, u_2) \cdot [(v_1, v_2) + (w_1, w_2)] \\ &= (u_1, u_2) \cdot [(v_1 + w_1, v_2 + w_2)] \\ &= u_1 (v_1 + w_1) + u_2 (v_2 + w_2) \\ &= \underbrace{u_1 v_1} + \underbrace{u_1 w_1} + \underbrace{u_2 v_2} + \underbrace{u_2 w_2} \\ &= \underbrace{u \cdot v} + \underbrace{u \cdot w} \end{aligned}$$
$\square$

Orthogonal Projection



The vector w is called the orthogonal projection of u onto a (Idea: Shadow)

we write $w = \text{proj}_a u$

Prop $\text{proj}_a u = \left(\dfrac{u \cdot a}{\|a\|^2}\right) a = \dfrac{(u \cdot a)}{\|a\|} \dfrac{a}{\|a\|}$

Corollary The vector v as shown, which is orthogonal to a is $u - w$

$= u - \text{proj}_a u$

$= u - \left(\dfrac{u \cdot a}{\|u\|^2}\right) a$

Proof of prop Clearly $\text{proj}_a u = ka$ for some $k \in \mathbb{R}$,

so we just need to find the magnitude of w.

Notice that $u - w$ is ortogonal to a

So $(u - w) \cdot a = 0$

$(u - ka) \cdot a = 0$

$u \cdot a - k a \cdot a = 0$

$k = \dfrac{u \cdot a}{a \cdot a} = \dfrac{u \cdot a}{\|a\|^2}$ $\square$

Examples 1) Find the angle between $(\sqrt{3}, 1)$ and $(\sqrt{3}, 3)$.

u v

$u \cdot v = \|u\| \cdot \|v\| \cos\theta$

$$6 = (2)(\sqrt{12})\cos\theta$$

$$\cos\theta = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{6}$$

(2) Find $\operatorname{proj}_v u = \left(\frac{u \cdot v}{\|v\|^2}\right) v = \frac{6}{12}(\sqrt{3}, 3) = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$

(3) Find The projection of u on the orthogonal of v.

$$= (\sqrt{3}, 1) - \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right) = \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$$

Distance from a point to a line in $\mathbb{R}^2$

Let $ax + by + c = 0$ (slope $-\frac{a}{b}$) be a line in $\mathbb{R}^2$ and let (x_0, y_0) be a point.

Claim First notice that the vector (a, b) (slope $\frac{b}{a}$) is orthogonal to the line!

proof Let (x, y) and (x_1, y_1) be on the line.

Then $(x - x_1, y - y_1)$ is parallel to the line

So $(a, b) \cdot (x - x_1, y - y_1)$

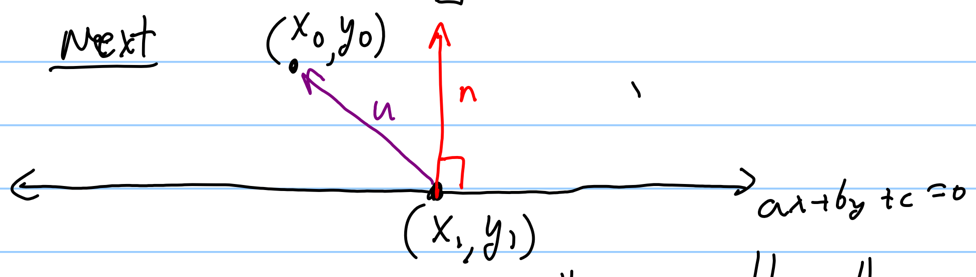
$$= ax - ax_1 + by - by_1$$

$$= ax + by - (ax_1 + by_1)$$

$$= (-c) - (-c)$$

$$= 0 \quad \square$$

Next



So we want to find: $\|\operatorname{proj}_n u\| = \|\operatorname{proj}_{(a,b)} u\|$

$$= \|\operatorname{proj}_{(a,b)} (x_0 - x_1, y_0 - y_1)\|$$

$$= \left\| \frac{(x_0 - x_1, y_0 - y_1) \cdot (a, b)}{\|(a, b)\|} \cdot \frac{(a, b)}{\|a, b\|} \right\|$$

magnitude 1

$$= \left| \frac{(x_0 - x_1, y_0 - y_1) \cdot (a, b)}{\|a, b\|} \right|$$

$$= \frac{|(ax_0 - ax_1 + by_0 - by_1)|}{\|(a,b)\|}$$

$$= \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \qquad \square$$