Vectors in $\mathbb{R}^3$

A vector in $\mathbb{R}^3$ looks like $u = (u_1, u_2, u_3)$.

For $n > 3$, our geometry will be complicated, but in $\mathbb{R}^3$ we can still do some.

Same as $\mathbb{R}^2$

- Basic Properties of addition and scalar multiplication.
- $\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$
- $d(u,v) = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2} = \|u - v\| = \|v - u\|$
- $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$

Prop $u \cdot v = \|u\| \cdot \|v\| \cos\theta$

- Same properties of dot product
- $\text{proj}_a u = \left(\frac{u \cdot a}{\|a\|^2}\right) a$

What makes vectors in $\mathbb{R}^3$ special is the cross product.

cross product

Define the special vectors $i = (1,0,0)$ $j = (0,1,0)$ $k = (0,0,1)$

Then, if $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$

define
$$u \times v = \det\begin{pmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$
$$= i(u_2 v_3 - u_3 v_2) - j(u_1 v_3 - v_1 u_3) + k(u_1 v_2 - u_2 v_1)$$
$$= (u_2 v_3 - u_3 v_2,\ v_1 u_3 - u_1 v_3,\ u_1 v_2 - v_1 u_2)$$

notice $i \times j = k$

$$j \times k = \det\begin{pmatrix} i & j & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = i$$

$j \times i = -k$

$k \times j = -i$

$i \times k = -j$

$k \times i = j$

Prop $a \times b = -(b \times a)$

Prop $a \times a = 0$

Think about these as determinants!

Prop $u \times v$ is orthogonal to u and to v

proof We want to show ① $(u \times v) \cdot u = 0$

And ② $(u \times v) \cdot v = 0$

① $(u \times v) \cdot u = \left(\det\begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix}, -\det\begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix}, \det\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right) \cdot u$

$= u_1 \det\begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} - u_2 \det\begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} + u_3 \det\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$

$= \det\begin{pmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = 0$

② similar

$\square$

N.B. Right-Hand Rule

Prop
(a) $u \times v = -(v \times u)$
(b) $u \times u = 0$
(c) $u \times 0 = 0$
(d) $u \times (v + w) = (u \times v) + (u \times w)$
(e) $(u + v) \times w = (u \times w) + (v \times w)$
(f) $c(u \times v) = (cu) \times v = u \times (cv)$ where $c \in \mathbb{R}$

Proof (f) $(cu) \times v = \det\begin{pmatrix} i & j & k \\ cu_1 & cu_2 & cu_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = c \det\begin{pmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = c(u \times v)$

(d) next time.

$\square$

Prop Let $\theta \in [0, \pi]$ be the angle between u & v. Then
(a) $\|u \times v\|^2 = \|u\|^2 \cdot \|v\|^2 - (u \cdot v)^2$
(b) $\|u \times v\| = \|u\| \cdot \|v\| \cdot \sin\theta$
(c) $\|u \times v\| =$ area of parallelogram with sides u & v.

Proof. (a) $\|u\times v\|^2 = (u_2v_3-u_3v_2)^2 + (u_1v_3-u_3v_1)^2 + (u_1v_2-v_1u_2)^2$

$$\begin{aligned} =\ & u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 \\ +\ & u_1^2v_3^2 - 2u_1u_3v_1v_3 + u_3^2v_1^2 \\ +\ & u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2 \end{aligned}$$

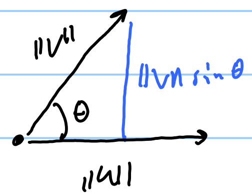
AND $\|u\|^2\|v\|^2 - (u\cdot v)^2 = (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2$

$$= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2$$
$$-(u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 + \text{Red}$$

(b) $\|u\times v\|^2 = \|u\|^2\|v\|^2 - (u\cdot v)^2 = \|u\|^2\|v\|^2 - \|u\|^2\|v\|^2\cos^2\theta$
$$= \|u\|^2\|v\|^2\sin^2\theta$$

Thus $\|u\times v\| = \|u\|\cdot\|v\|\cdot\sin\theta$ (since $\theta\in[0,\pi]$, $\sin\theta\geq 0$)

(c)



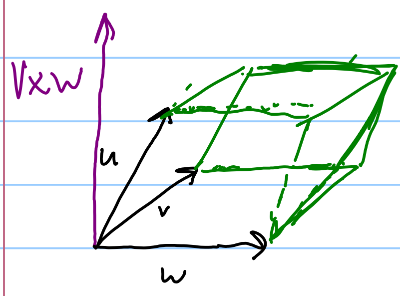
Area of parallelogram $= \|u\|\,\|v\|\sin\theta$
$= \|u\times v\|$

□

Corollary: If u, v are orthogonal then $\|u\times v\| = \|u\|\cdot\|v\|$

Scalar Triple Product

How can we find the volume of the parallelepiped with edges u, v, w? (Assuming that u, v, w don't all lie in a plane)



Area of base: $\|v\times w\|$

Height: $\|\operatorname{proj}_{(v\times w)} u\|$

$$= \left\| \frac{u\cdot(v\times w)}{\|(v\times w)\|^2}(v\times w) \right\|$$

$$= \left\| \overbrace{\frac{u\cdot(v\times w)}{\|v\times w\|}}^{c} \cdot \overbrace{\frac{v\times w}{\|v\times w\|}}^{a} \right\| \qquad \|ca\| = |c|\,\|a\|$$

$$= \left| \frac{u\cdot(v\times w)}{\|v\times w\|} \right|$$

Therefore volume is $|u\cdot(v\times w)|$

<u>Defn</u> $|u\cdot(v\times w)|$ is called the scalar triple product of u, v, w.

<u>Prop</u> 1 Since the volume is the same, $|u\cdot(v\times w)| = |u\cdot(w\times v)| = |v\cdot(u\times w)|$

$= |v\cdot(w\times u)| = |w\cdot(u\times v)| = |w\cdot(v\times u)|$

2 Furthermore, $|u\cdot(v\times w)| = 0$ iff u, v, w are coplanar.

Also $u\cdot(v\times w) = u\cdot\left(\det\begin{pmatrix} v_2 & v_3 \\ w_2 & w_3\end{pmatrix}, -\det\begin{pmatrix} v_1 & v_3 \\ w_1 & w_3\end{pmatrix}, \det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2\end{pmatrix}\right)$

$= u_1\det\begin{pmatrix} v_2 & v_3 \\ w_2 & w_3\end{pmatrix} - u_2\det\begin{pmatrix} v_1 & v_3 \\ w_1 & w_3\end{pmatrix} + u_3\det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2\end{pmatrix}$

$= \det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3\end{pmatrix}$

<u>Example</u> Find the volume of the parallelepiped formed by the vectors $u = (1,0,1)$ $v = (2,1,4)$ $w = (0,1,1)$

<u>soln</u>

$\det\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 0 & 1 & 1\end{pmatrix} = 1\cdot\det\begin{pmatrix} 1 & 4 \\ 1 & 1\end{pmatrix} + 1\det\begin{pmatrix} 2 & 1 \\ 0 & 1\end{pmatrix}$

$= -3 + 2$

$= -1$

<u>Lines and Planes in $\mathbb{R}^3$</u>

Suppose we have a plane in $\mathbb{R}^3$.

Let (x_1, y_1, z_1) be a fixed point on this plane

Let $n = (a,b,c)$ be a vector perpendicular to the plane.

Then if (x,y,z) is in the plane,

$(a,b,c)\cdot(x-x_1, y-y_1, z-z_1) = 0$

$\Rightarrow ax - ax_1 + by - by_1 + cz - cz_1 = 0$

$\Rightarrow ax + by + cz + d = 0$ (where $d = -ax_1 - by_1 - cz_1$)

So this is an equation for the plane.

Example Find an equation of the plane perpendicular to $n = (3, 2, -1)$ which contains $(0, 1, 3)$

Soln $3x + 2y - z + d = 0$

use $(0,1,3)$ to solve for d

$0 + 2 - 3 + d = 0$

$d = 1$

Eqn $3x + 2y - z + 1 = 0$

Example 2 Find an equation for the plane containing $(1,2,-3)$, $(2,-3,4)$, $(-3,1,2)$

Vectors in the plane: $(-3,1,2) - (1,2,-3) = (-4,-1,5)$

$(-3,1,2) - (2,-3,4) = (-5,4,-2)$

so let $n = (-4,-1,5) \times (-5,4,-2) = (-18, -33, -21)$

... $\square$

(Parametric) Equation of a line containing (x_1, x_2, x_3) (y_1, y_2, y_3)

$(x_1, x_2, x_3) + t(y_1 - x_1, y_2 - x_2, y_3 - x_3)$ for all $t \in \mathbb{R}$

Another way to find eqn of a plane If x is in the plane containing a, b, c then $(x-a) \cdot ((x-b) \times (x-c)) = 0$

$$\Rightarrow \det \begin{pmatrix} x_1 - a_1 & x_2 - a_2 & x_3 - a_3 \\ x_1 - b_1 & x_2 - b_2 & x_3 - b_3 \\ x_1 - c_1 & x_2 - c_2 & x_3 - c_3 \end{pmatrix} = 0$$

Finally Similar to $\mathbb{R}^2$:

If (x_0, y_0, z_0) is a point, and $ax + by + cz + d = 0$ is a plane

then the distance is $$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Proof omitted.

// end of ch 4