Chapter 5 Vector Spaces.

Defn A real vector space V (or a vector space V over $\mathbb{R}$) is a set of objects called "vectors" together with 2 binary operations, vector addition and scalar multiplication, all satisfying the following properties:

If $v, w, u \in V$ and $a, b, c \in \mathbb{R}$ then

① If $u+v \in V$ (closure under addition)

② $(u+v)+w = u+(v+w)$

③ $u+v = v+u$

④ There is an element $0 \in V$ so that $u+0 = u = 0+u$ for all $u \in V$

(The zero vector = The additive identity)

⑤ $\forall u \in V \quad \exists v \in V$ so that $u+v = 0$ ((additive) Inverses Exist)

Convention: we call $v = -u$

⑥ $au \in V$ (closure under scalar mult)

⑦ $a(u+v) = au + av$

⑧ $a(bu) = (ab)u$

⑨ $(a+b)u = au + bu$

⑩ $1u = u$

In the context of a real vector space, we call $a, b, c \in \mathbb{R}$ scalars

Examples

① $\mathbb{R}^2$

② $\mathbb{R}^3$

③ $\mathbb{R}^n$ for any $n \geq 1$

④ The set of all 4×5 matrices.

⑤ The set of all functions from $\mathbb{R}$ to $\mathbb{R}$,

where we define $(f+g)(x) = f(x) + g(x)$

$(cf)(x) = c \cdot f(x)$

⑥ The set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

⑦ The set of all polynomials with real coefficients

NON EXAMPLE The set of all polynomials with rational coefficients.

For example $(\pi)(x+1) = \pi x + \pi$ is not a polynomial with Rational coefficients.

So property 6 fails.

⑧ The set of all polynomials of degree ≤ 4.

NON Example The set of all polynomials of degree 4.

For example $(x^4+2x) + (-x^4+x^3) = x^3+2x$

fails properties 1, 3, 6

However, these are just examples, let's prove some stuff.

proposition $\forall u \in V$, $\forall c \in \mathbb{R}$:

ⓐ $0 \cdot u = 0$

ⓑ $c \cdot 0 = 0$

ⓒ $(-1)u = -u$

ⓓ If $cu=0$ then $\overset{\text{scalar}}{c}=0$ or $\overset{\text{vector}}{u}=0$

proof ⓐ $0u + 0u = (0+0)u = 0u$

Thus $0u + 0u = 0u$

$0u + \underbrace{0u + (-0u)} = \underbrace{0u + (-0u)}$

$\underbrace{0u + 0} = 0$

$0u = 0$

ⓑ $c0 + c0 = c(0+0) = c0$

Then $c0 + c0 + (-c0) = c0 + (-c0)$

$\Rightarrow c0 + 0 = 0$

$\Rightarrow c0 = 0$

ⓒ $(-1)\cdot u = (-1)\cdot u + 0$

$= (-1)\cdot u + (u + (-u))$

$= ((-1)\cdot u + u) + (-u)$

$= ((-1)\cdot u + 1\cdot u) + (-u)$

$= (-1+1)\cdot u + (-u)$

$= 0\cdot u + (-u)$

$= 0 + (-u)$

$= -u$

ⓓ Assum $cu=0$. Suppose $c \neq 0$

Then
$$\begin{aligned} u &= 1u \\ &= \left(\tfrac{1}{c}\cdot c\right)u \\ &= \tfrac{1}{c}(cu) \\ &= \tfrac{1}{c}\cdot 0 \quad \text{by assumption} \\ &= 0 \quad \text{by part b} \ \square \end{aligned}$$

Lemma If $v+u=0$ and $w+u=0$, then $v=w$.

proof If $v+u=0$ and $w+u=0$, then

$$\begin{aligned} v+u &= w+u \\ (v+u)+(-u) &= (w+u)+(-u) \\ v+(u+(-u)) &= w+(u+(-u)) \\ v+0 &= w+0 \\ v &= w \ \square \end{aligned}$$

Lemma If $v+u=0$ and $u+w=0$, then $v=w$.

pf
$$\begin{aligned} (v+u)+w &= v+(u+w) \\ 0+w &= v+0 \\ w &= v \ \square \end{aligned}$$

(Vector) Subspaces

Definition Suppose V is a vector space over $\mathbb{R}$. A subset $W \subseteq V$ is called a (vector) subspace if

① $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$

AND ② $w \in W$ and $c \in \mathbb{R} \Rightarrow cw \in W$

Lemma If $W \subseteq V$ is a subspace and if $w \in W$, then

Ⓐ $0 \in W$

Ⓑ $-w \in W$

proof Ⓐ $w \in W \overset{②}{\Rightarrow} 0w \in W \overset{\text{Prop, part a}}{\Rightarrow} 0 \in W$

Ⓑ $w \in W \overset{②}{\Rightarrow} (-1)\cdot w \in W \underset{\text{Prop, part c}}{\Rightarrow} -w \in W$

$\square$

Examples: ① Suppose $V = \mathbb{R}^2$

Then the following are subspaces of V: $\mathbb{R}^2$

$\{0\}$

any line through the origin

② Suppose $V = \mathbb{R}^3$. Subspaces: $\mathbb{R}^3$

$\{0\}$

any line through the origin

check ← any plane through the origin

Let $ax+by+cz=0$ be a plane through the origin in $\mathbb{R}^3$

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be points on this plane

Then $(x_1,y_1,z_1)+(x_2,y_2,z_2) = (x_1+x_2,\ y_1+y_2,\ z_1+z_2)$

And $a(x_1+x_2)+b(y_1+y_2)+c(z_1+z_2) = \underline{ax_1}+\underline{ax_2}+\underline{by_1}+\underline{by_2}+\underline{cz_1}+\underline{cz_2}$

$= 0 + 0$

$= 0$

Next let $m \in \mathbb{R}$ and (x,y,z) be on the plane

Then $m(x,y,z) = (mx, my, mz)$

and $a(mx)+b(my)+c(mz) = m(ax+by+cz) = m\,0 = 0$ □

③ If $V = M_{3\times3}$ = the set of all 3×3 real matrices, then the following are subspaces:

ⓐ All 3×3 matrices with middle entry 0.

ⓑ The set of all matrices of the form αI for $\alpha \in \mathbb{R}$

Null Space of A

④ If A is a particular $n\times n$ matrix, then the set of solutions to $Ax=0$ is a subspace of $\mathbb{R}^n$

proof Ⓐ Assume $y_1, y_2 \in \mathbb{R}^n$ are both solutions to $Ax=0$

Then $A(y_1+y_2) = Ay_1 + Ay_2 = 0+0 = 0$

Ⓑ Let $c \in \mathbb{R}$, and assume $y \in \mathbb{R}^n$ is a solution to $Ax=0$

Then $A(cy) = c(Ay) = c\cdot 0 = 0$ □

⑤ $\{0\} \subseteq \ldots$ Real Polynomials of degree ≤ 100 $\subseteq$ Polynomials with real coefficients $\subseteq$ differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ $\subseteq$ continuous functions from $\mathbb{R}$ to $\mathbb{R}$ $\subseteq$ functions from $\mathbb{R}$ to $\mathbb{R}$

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Prop If W_1 and W_2 are subspaces of V, then so is $W_1 \cap W_2$

Exercise Prove this **and** give a counter-example for $W_1 \cup W_2$.