Linear Combination / Span

Definition A linear combination of the vectors $v_1, \dots, v_k$ is any vector of the form $c_1 v_1 + c_2 v_2 + \dots + c_k v_k$ where each of the c's are scalars.

Example 1 Let $V = \mathbb{R}^3$. Consider $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$.
Then any vector in $\mathbb{R}^3$ is a linear combination of i, j, k.
Any Example $(u_1, u_2, u_3) = u_1 i + u_2 j + u_3 k$

Example 2 Is $(-4, 5, -17, 16)$ a linear combination of $(1,4,2,6)$ and $(2,1,7,-1)$?
Yes $(-4, 5, -17, 16) = 2(1,4,2,6) - 3(2,1,7,-1)$

Example 3 Is $(1,2,4)$ a linear combination of $(1,0,1)$ and $(0,1,1)$?
NO **Proof** Suppose $(1,2,4) = c_1(1,0,1) + c_2(0,1,1)$
Then
$$1 = c_1$$
$$2 = c_2$$
$$4 = c_1 + c_2$$
No solution.

Defn Suppose $v_1, \dots, v_k \in V$ over $\mathbb{R}$
Then $\text{span}\{v_1, \dots, v_k\} = \{c_1 v_1 + c_2 v_2 + \dots + c_k v_k \mid c_1, \dots, c_k \in \mathbb{R}\}$
$=$ the set of linear combinations of $v_1, \dots, v_k$

Then $\text{span}\{v_1 \dots v_k\} = V$ iff every vector in V can be expressed as a linear combination of $v_1 \dots v_k$

In example 1, $\text{span}(i,j,k) = \mathbb{R}$
In example 3, $\text{span}\{(1,0,1),(0,1,1)\}$ = the plane containing $(0,0,0), (1,0,1) (0,1,1)$.
and $(1,2,4)$ is not on this plane.

Prop Suppose $v_1 \dots v_k \in V$ over $\mathbb{R}$.

① $\text{span}\{v_1, \dots v_k\}$ is a subspace of V

② If $v_1 \dots v_k \in W$ where $W \subseteq V$ and W is a subspace of V, Then $\text{span}\{v_1 \dots v_k\} \subseteq W$

proof ① Suppose $u_1, u_2 \in \text{Span}\{v_1 \dots v_k\}$.

Then there are $a_1, \dots a_k, b_1 \dots, b_k \in \mathbb{R}$

so that $u_1 = a_1 v_1 + a_2 v_2 + \dots a_k v_k$.

and $u_2 = b_1 v_1 + b_2 v_2 + \dots b_k v_k$

Then $u_1 + u_2 = (a_1 v_1 + a_2 v_2 + \dots + a_k v_k) + (b_1 v_1 + b_2 v_2 + \dots + b_k v_k)$

$= (a_1 + b_1) v_1 + (a_2 + b_2) v_2 + \dots + (a_k + b_k) v_k$

Thus the span is closed under addition.

Now suppose $\alpha \in \mathbb{R}$. Then $\alpha u_1 = \alpha(a_1 v_1 + a_2 v_2 + \dots + a_k v_k)$

$(\alpha a_1) v_1 + (\alpha a_2) v_2 + \dots + (\alpha a_k) v_k$

Thus it is also closed under scalar multiplication.

② Suppose $v_1, \dots v_k \in W$ and W is a subspace of V.

Suppose $x \in \text{Span}\{v_1, \dots v_k\}$

Then there are $c_1 \dots c_k \in \mathbb{R}$ so that

$x = c_1 v_1 + \dots + c_k v_k$

Then since $v_1 \dots v_k \in W$ and W is closed under scalar mult,

$c_1 v_1, \dots, c_k v_k \in W$.

Then since W is closed under addition,

$c_1 v_1 + \dots + c_k v_k \in W$. $\square$

Linear Independence

Consider $i, j, k, (0,1,1)$ as vectors in $\mathbb{R}^3$.

Then $\text{span}\{i, j, k, (0,1,1)\} = \mathbb{R}^3$

But this is kind of redundant

① Because $\text{span}\{i, j, k, (0,1,1)\} = \text{span}\{i, j, k\} = \text{span}\{i, j, (0,1,1)\}$ " $\text{span}\{i, k, (0,1,1)\}$

② A vector in $\mathbb{R}^3$ is not uniquely expressible as a linear combination of $i, j, k, (0,1,1)$

To illustrate $(4,7,2) = 4i + 5j + 0k + 2(0,1,1)$

$(4,7,2) = 4i + 7j + 2k + 0(0,1,1)$

Definition Suppose $v_1 \ldots v_k \in V$ over $\mathbb{R}$.

(a) $\{v_1, \ldots, v_k\}$ is linearly dependent if there exist $c_1 \ldots c_k \in \mathbb{R}$ so that

$$c_1 v_1 + \ldots + c_k v_k = 0$$

(where not all of the c_i's are 0)

(b) We say $\{v_1, \ldots, v_k\}$ is linearly independent if the only solution to $c_1 v_1 + \ldots + c_k v_k = 0$ is

$$c_1 = c_2 = \ldots = c_k = 0$$

Example A $(1,1,0), (0,1,1), (2,2,2)$ are linearly independent.

Proof Suppose there exist $c_1, c_2, c_3 \in \mathbb{R}$ so that

$$c_1(1,1,0) + c_2(0,1,1) + c_3(2,2,2) = (0,0,0).$$

Then
$$c_1 + 2c_3 = 0$$
$$c_1 + c_2 + 2c_3 = 0$$
$$c_2 + 2c_3 = 0$$

... Gauss Elimination ... then $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

Lemma If $v_1 \ldots v_k$ are linearly independent and $u \in \text{span}\{v_1 \ldots v_k\}$ then u is uniquely expressible as a lin comb of $v_1 \ldots v_k$.

proof Suppose $\exists\, a_1 \ldots a_k, b_1 \ldots b_k \in \mathbb{R}$

and $u = a_1 v_1 + \ldots + a_k v_k$

and $u = b_1 v_1 + \ldots + b_k v_k$

Subtracting the equations

$$0 = (a_1 - b_1) v_1 + \ldots + (a_k - b_k) v_k$$

Then since $\{v_1, \ldots, v_k\}$ is linearly independent,

$$\left.\begin{array}{l} a_1 - b_1 = 0 \\ a_2 - b_2 = 0 \\ \vdots \\ a_k - b_k = 0 \end{array}\right\} \Rightarrow \begin{array}{l} a_1 = b_1 \\ a_2 = b_2 \\ \vdots \\ a_k = b_k \end{array} \quad \square$$

Example B

Show $\{(1,2,1), (2,2,-1), (4,6,1)\}$ is linearly dependent.

pf $2(1,2,1) + (2,2,-1) - (4,6,1) = (0,0,0)$ $(*)$

Notice that we have $(4,6,1) = 2(1,2,1) + (2,2,-1)$

So let's look at

$$\det\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \\ 4 & 6 & 1 \end{pmatrix} \underset{R_3 - 2R_1}{=} \det\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \\ 2 & 2 & -1 \end{pmatrix} \underset{R_3 - R_2}{=} \det\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Hmmm.

But what about this matrix?

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 2 & 6 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (*) \quad \text{Same equation}$$

(And since this has a non-trivial solution, $\det\begin{pmatrix} 1 & 2 & 4 \\ 2 & 2 & 6 \\ 1 & -1 & 1 \end{pmatrix} = 0$)

Lemma If $v_1 \ldots v_n \in \mathbb{R}^n$ then

$\{v_1 \ldots v_n\}$ is linearly independent $\Leftrightarrow \det(v_1 \ldots v_n) \neq 0$

proof $\{v_1 \ldots v_n\}$ is linearly independent

$\Leftrightarrow (v_1 \ldots v_n)\, x = 0$ has no non trivial solutions

$\Leftrightarrow \det(v_1 \ldots v_n) \neq 0$ $\square$

Add "columns are lin ind" to list of invertibility conditions.

Prop Suppose $v_1 \ldots v_k \in \mathbb{R}^n$ and $k > n$.

Then $\{v_1 \ldots v_k\}$ is linearly dependent.

proof Examine $c_1 v_1 + \ldots + c_k v_k = 0$

$$\begin{pmatrix} v_1 & \cdots & v_k \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since $k > n$ there are more variables than equations.

Hence, this has ∞-many solutions, in particular, a non-trivial one.

so $\{v_1 \ldots v_k\}$ is linearly dependent $\square$

Examples $(\mathbb{R}^2)$ $\quad c_1 v_1 + c_2 v_2 = 0$

$$\Leftrightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{where} \quad v_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \quad v_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

So there is a NON trivial solution

iff v_1, v_2 are lin ind

iff $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0$

iff $a_{11} a_{22} \neq a_{12} a_{21}$

iff $\dfrac{a_{11}}{a_{12}} \neq \dfrac{a_{21}}{a_{22}}$

iff v_1, v_2 are not multiples of each other

iff the area of the parallelogram is not 0.

iff v_1, v_2 aren't on a line with the origin

$(\mathbb{R}^3)$ similarly $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$\Leftrightarrow \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$

So $\{v_1, v_2, v_3\}$ is lin ind $\Leftrightarrow \det \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \neq 0$

$\Leftrightarrow \det \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \neq 0$

$\Leftrightarrow$ scalar triple product $\neq 0$

$\Leftrightarrow$ volume of parallelepiped $\neq 0$

$\Leftrightarrow$ v_1, v_2, v_3 aren't coplanar with the origin

Basis + Dimension

Definition Suppose $v_1 \dots v_k \in V$ over $\mathbb{R}$.

we say $\{v_1 \dots v_k\}$ is a basis for V if

1. $\{v_1 \dots v_k\}$ is lin independent
2. $\operatorname{span}\{v_1 \dots v_k\} = V$

(Informally, Basis $\equiv$ smallest set which spans V

$\equiv$ largest lin ind set in V.)

We've already seen that $\{v_1 \dots v_k\}$ is a basis of V if every vector in V can be uniquely expressed as a lin comb of $\{v_1 \dots v_k\}$.