Recall the following definition.

Defn If V is a vector space over $\mathbb{R}$, then a set of vectors $\{v_1, \dots, v_k\}$ forms a basis for V if

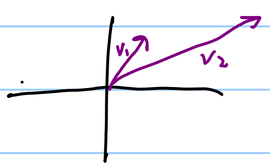
① $\{v_1, \dots, v_k\}$ is linearly independent

② span $\{v_1, \dots, v_k\}$.

We've already seen that this is equivalent to "every vector in V can be uniquely expressed as a linear combination of $\{v_1, \dots, v_k\}$"

Example $\{(1,0), (0,1)\}$ is a basis for $\mathbb{R}^2$

Any set of 2 lin ind vectors forms a basis for $\mathbb{R}^2$



Definition A vector space is finite-dimensional if it has a finite basis

Examples Ⓐ $\mathbb{R}^n$ is finite dimensional

The standard basis is $\{e_i\}_{i=1}^{n}$ where e_i is all zeros, except for a 1 in the i^{th} coordinate.

Ⓑ The set of polynomials over $\mathbb{R}$ is not finite dimensional.

pf If $\{f_1, \dots, f_n\}$ is a finite set of polynomials, then any linear combination of them will have degree less than or equal to the maximum degree of $f_1, \dots, f_n$.

However, there is an infinite basis: $\{1, x, x^2, x^3, x^4, \dots\}$

Ⓒ The set of all polynomials of degree ≤ 4 is finite dimensional

Basis $\{1, x, x^2, x^3, x^4\}$

Lemma 1 Suppose $\{v_1, \ldots, v_n\}$ is a basis for the vector space V over $\mathbb{R}$.

Suppose $u_1, \ldots, u_k \in V$ and $k > n$.

Then $u_1, \ldots, u_k$ are linearly dependent.

Corollary Let V be a finite dimensional vector space. Then any two bases for V have the same number of vectors.

We call this number the dimension of V

Proof of Lemma 1 Let $\{v_1, \ldots, v_n\}$ be a basis for V. Let $u_1, \ldots, u_k \in V$ with $k > n$.

Since $\text{span}\{v_1, \ldots, v_n\} = V$, then there are a's $\in \mathbb{R}$ so that

$$u_1 = a_{11} v_1 + a_{21} v_2 + \ldots + a_{n1} v_n$$

$$\vdots$$

$$u_k = a_{1k} v_1 + a_{2k} v_2 + \ldots + a_{nk} v_n$$

So now consider scalars $c_1, \ldots c_k \in \mathbb{R}$ so that

$$c_1 u_1 + c_2 u_2 + \ldots + c_k u_k = 0$$

$$\Rightarrow c_1 (a_{11} v_1 + a_{21} v_2 + \ldots + a_{n1} v_n) + \ldots + c_k (a_{1k} v_1 + a_{2k} v_2 + \ldots + a_{nk} v_n) = 0$$

$$\Rightarrow (c_1 a_{11} + c_2 a_{12} + \ldots + c_k a_{1k}) v_1 + \ldots + (c_1 a_{n1} + c_2 a_{n2} + \ldots + c_n a_{nk}) v_n = 0$$

But since $\{v_1, \ldots, v_n\}$ is lin ind, each of the coefficients is zero.

$$\Rightarrow \begin{array}{c} c_1 a_{11} + c_2 a_{12} + \ldots + c_k a_{1k} = 0 \\ \vdots \\ c_1 a_{n1} + c_2 a_{n2} + \ldots + c_k a_{nk} = 0 \end{array}$$

$$\Rightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = 0$$

So this is a Chapter 1 problem with more variables than equations. Hence there are ∞-many solutions. In particular, there is a non trivial solution. Then this (not all 0) sequence of $c_1, \ldots, c_k$ also sets

$$\sum_{h=1}^{k} c_h u_h = c_1 u_1 + \ldots + c_k u_k = 0 \quad \square$$

So if V has dimension n, then a basis for V must be n linearly independent vectors. But does the converse hold?

That is, if V has dimension n, then is any set of n linearly independent vectors also a basis? (Yes...)

Lemma 2 Suppose V is a finite dimensional vector space over $\mathbb{R}$. Then any finite set of linearly independent vectors can be expanded to a basis for V.

Corollary If V is n-dimensional, then a set of n lin ind vectors is a basis already.

proof of cor Suppose $\{v_1, \ldots, v_n\}$ are lin ind. By lemma 2, they can be expanded to a basis. But by lemma 1, if we add any vectors, the set won't be lin ind. Thus it must be a basis to start with $\square$

Proof of Lemma Let $S = \{v_1, \ldots, v_k\} \subseteq V$ be a set of linearly independent vectors.

Case I Span $\{v_1, \ldots v_k\} = V$. Then we are done.

Case II Span $\{v_1, \ldots, v_k\} \neq V$. Then there is some $u \in V - \text{span}\{v_1, \ldots v_k\}$

Claim $\{v_1, \ldots, v_k, u\}$ is lin ind.

pf If not then $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k + cu = 0$ where not all of the c_j are 0.

If $c = 0$, this contradicts that $\{v_1, \ldots, v_k\}$ is lin ind.

If $c \neq 0$, then

$$u = \left(-\frac{1}{c}\right)\left[c_1 v_1 + c_2 v_2 + \cdots + c_k v_k\right]$$
$$= -\frac{c_1}{c} v_1 - \frac{c_2}{c} v_2 - \cdots - \frac{c_k}{c} v_k$$

which contradicts that $u \notin \text{span}\{v_1, \ldots, v_k\}$ $\square$

Then if this new set, still does not span V, we can repeat the argument until we reach $\dim V$ vectors. By lemma 1, we can't go farther than this.

$\square$

Example $\{(2,1,-1), (0,1,3), (4,3,1)\}$ is lin dep:

$$2(2,1,-1) + (0,1,3) - (4,3,1) = 0$$

But $\{(2,1,-1), (0,1,3)\}$ is lin ind

AND $\{(2,1,-1), (0,1,3), (1,0,0)\}$ is a basis.

Chapter 6 Rank & Nullity.

not nec. square.

There are 3 important vector spaces associated with any matrix

Def Let A be an $m \times n$ matrix.

Let $r_1, \dots, r_m$ be the row vectors of A

Let $c_1, \dots, c_n$ be the column vectors of A.

(1) $\text{span}\{r_1, \dots, r_m\}$ is a subspace of $\mathbb{R}^n$ and is called the Row Space of A

(2) $\text{span}\{c_1, \dots, c_n\}$ is a subspace of $\mathbb{R}^m$ and is called the column space of A.

(3) The set of solutions to $Ax = 0$ is a subspace of $\mathbb{R}^n$ and is called the Null space of A

Previously proved

Example Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$

Row space(A) = $\text{span}\{(1,2,1), (-1,1,0), (0,3,1)\}$

$= \text{span}\{(1,2,1), (-1,1,0)\}$

which has dimension 2

column space(A) = $\text{span}\{(1,-1,0), (2,1,3), (1,0,1)\}$

$= \text{span}\{(1,-1,0), (1,0,1)\}$

which has dimension 2

To find null space(A), we solve $Ax = 0$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right) \quad R_1 + R_2 \to R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right) \quad R_2 - R_3 \to R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Back solve

Let $x_3 = t$

Row 2 $3x_2 + t = 0$

$x_2 = \frac{-t}{3}$

Row 1 $x_1 - 2\frac{t}{3} + t = 0$

$x_1 = -\frac{t}{3}$

General solution

$(x_1, x_2, x_3) = \left(-\frac{t}{3}, \frac{-t}{3}, t\right) = t\left(-\frac{1}{3}, -\frac{1}{3}, 1\right)$

So null space(A) = $\text{span}\{(-\frac{1}{3}, -\frac{1}{3}, 1)\}$

which has dimension 1

Target Theorem 1 For any matrix A, dim row space (A) = dim column space (A)

Defn This common dimension is called the RANK of A.

Target theorem 2 If A is an $m \times n$ matrix, then

$$\text{Rank}(A) + \dim \text{null space}(A) = n.$$

Rank-Nullity Theorem.

Lemma Suppose $B \sim A$. Then row space (B) = row space of (A)

proof Each row of B is a lin comb of the rows of A.

Thus row space $(B) \subseteq$ Row space (A)

Also vice versa $\square$

Example Let

$$A = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & 2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{pmatrix} \sim \begin{array}{cl} \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} \leftarrow r_1 \\ \leftarrow r_2 \\ \leftarrow r_3 \\ \end{array} \end{array}$$

Then Row space = span $\{r_1, r_2, r_3\}$ which has dimension 3.