Example from last time

$$A = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & 2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{pmatrix} \xrightarrow{REF} \sim \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus row space $(A) = \text{Span}\{(1,3,-5,1,5), (0,1,-2,2,-7), (0,0,0,1,-5)\}$

This method always works, because

Exercise (To hand in)

The non-zero rows of a matrix in REF are linearly independent.

Surprisingly, the same method (row operations) gives us a way to find a basis for the column space of A.

Lemma Suppose $B \sim A$. Then a set of columns in A is linearly independent if and only if the corresponding set of columns of B is linearly independent.
(eg cols 1,3,4 of A are lin ind $\Leftrightarrow$ cols 1,3,4 of B are lin ind)

Proof Let A^* be the submatrix of A made up of the columns in question, and let B^* be the submatrix of B made up of the corresponding columns of B.

eg If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{pmatrix}$ and if we are looking at columns 1,3,4,

then $A^* = \begin{pmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \end{pmatrix}$

The point is that whatever sequence of row operations takes A to B will also take A^* to B^*, and so $A^* \sim B^*$.

So now the columns in question (eg 1,3,4) of A are lin ind

$a_{11}x_1 + a_{12}x_2 = 0$

$a_{21}x_1 + a_{22}x_2 = 0$

$\Leftrightarrow$ the columns of A^* are lin ind

$\Leftrightarrow$ $A^*x = 0$ has only the trivial solution

$\Leftrightarrow$ $B^*x = 0$ has only the trivial solution

$\Leftrightarrow$ the colums of B^* are lin ind

$\Leftrightarrow$ the columns in question of B (e.g. 1,3,4) are lin ind $\square$

How about the same example?

$$\begin{array}{cc} & \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ A = & \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & 2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{pmatrix} \end{array} \xrightarrow{REF} \sim \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} = B$$

So, since $\operatorname{colspace}(B) = \operatorname{span}\{u_1, u_2, u_4\}$,

then $\operatorname{colspace}(A) = \operatorname{span}\{v_1, v_2, v_4\}$.

Target Theorem 4 For any matrix A, $\dim \operatorname{colspace}(A) = \dim \operatorname{Rowspace}(A)$

pf Let $B \sim A$ so that B is in (R)REF

Then $\dim \text{row space}(A) = \dim \text{row space}(B) = \#$ of non zero rows of B

AND $\dim \text{col space}(A) = \dim \text{col space}(B) = \#$ of pivot entries of B $(*)$

Since B is in REF, $(*)$ are the same $\square$

Definition We call this common dimension the Rank of A.

In both of our examples, we saw that the following was true:

Rank-Nullity Theorem $\dim \text{null space}(A) + \text{Rank of}(A) = \#$ of columns.

Idea of proof Reduce A to RREF, and call this B

We already know that $\operatorname{rank} A = \operatorname{rank} B = \#$ of pivot columns of B.

Also $\dim \text{null space}(A) = \dim \text{null space}(B) = \#$ of non pivot columns

$= \#$ of parameters. $\square$ Sketch

Same example one more time:

$$A = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & 2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{pmatrix} \xrightarrow{\text{REF}} \sim \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad R_1 - 3R_2$$

$$\begin{pmatrix} 1 & 0 & 1 & -5 & 26 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} R_1 + 5R_3 \\ R_2 - 2R_3 \end{matrix}$$

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} & & t_2 & & t_1 \end{matrix} \end{array}$$

General solution to $Ax = \cancel{0}\ b$ is

$$(x_1, x_2, x_3, x_4, x_5) = (-t_1 - t_2 + 1,\ 2t_2 - 3t_1 + 5,\ t_2 + 6,\ 5t_1 - 3,\ t_1 + \pi)$$

$$= t_1(-1, -3, 0, 5, 1) + t_2(-1, 2, 1, 0, 0) + (1, 5, 6, -3, \pi)$$

__Proposition__ Suppose that A is a matrix and that x_0 is __any__ solution to $Ax = b$. Suppose that $\{v_1, \dots, v_k\}$ is a basis for the null space of A. Then every solution to $Ax = b$ is of the form

$$x = x_0 + c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$

(where $c_1, \dots, c_k \in \mathbb{R}$)

Furthermore, all of these are solutions to $Ax = b$.

__proof__ Fix a solution to $Ax = b$, and call it x_0.

Then if x is __any__ other solution to $Ax = b$, we have

$$A(x - x_0) = Ax - Ax_0 = b - b = 0$$

In other words $x - x_0$ is in the null space of A. So there are $c_1, \dots, c_k \in \mathbb{R}$ so that

$$x - x_0 = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$

$$x = x_0 + c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$

OTOH, we nee to show that every

$x_0 + c_1 v_1 + c_2 v_2 + \dots + c_k v_k$ is a solution..

$$
\begin{aligned}
&A(x_0 + c_1 v_1 + c_2 v_2 + \dots + c_k v_k) \\
&= Ax_0 + A(c_1 v_1) + A(c_2 v_2) + \dots + A(c_k v_k) \\
&= Ax_0 + c_1(Av_1) + c_2(Av_2) + \dots + c_k(Av_k) \\
&= b + c_1(0) + c_2(0) + \dots + c_k(0) \\
&= b \quad \square
\end{aligned}
$$

Theorem Let A be an $n \times n$ Matrix. TFAE

(a) A^{-1} exists

(b) $\det A \neq 0$

(c) $A \sim I_n$

(d) $Ax = 0$ has only the trivial solution

(e) For every b, $Ax = b$ has the unique solution $x = A^{-1}b$

(f) The Rows of A are linearly independent

(g) The colums of A are linearly independent

(h) A has Rank n

Rank-Nullity Theorem

Target Theorem 1

Chapter 7 Eigenvalues & Eigenvectors

In chapters 7+8, we will consider square matrices as they act as functions from vectors to vectors.

For example $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ is a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ that takes

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 \\ 7 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\text{and} \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Notice that these last two are equations of the form $Av = \lambda v$ where λ is a scalar. It turns out that these vectors have special significance + many applications.

Defn Suppose A is an $n \times n$ matrix, λ a scalar, and $v \in \mathbb{R}^n$ a vector, (if $v \neq 0$) and $Av = \lambda v$

Then v is called an eigenvector of A
and λ is called an eigenvalue of A.

First, notice that in our example $u_1 = \binom{-1}{1}$ and $u_2 = \binom{1}{2}$ are lin ind, so they form a basis for $\mathbb{R}^2$. Thus, for any other vector $u \in \mathbb{R}^2$, there are scalars $c_1, c_2 \in \mathbb{R}$ so that $u = c_1 u_1 + c_2 u_2$

so $Au = A(c_1 u_1 + c_2 u_2) = c_1 Au_1 + c_2 Au_2$

$= -c_1 u_1 + 5 c_2 u_2$

So this is a "simplified" version of the function Au

Next if v is an eigenvector of A (with λ the corresponding eigenvalue) and if $c \neq 0$ is a scalar, then cv is also an eigenvector of A (with the same eigenvalue).

pf Suppose $Av = \lambda v$. Then $A(cv) = c(Av) = c\lambda v = \lambda(cv)$ $\square$

In our example above: $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 15 \\ 30 \end{bmatrix} = 5\begin{bmatrix} 3 \\ 6 \end{bmatrix}$

Now If A is a square matrix, how can we find its eigenvalues?

Suppose $Av = \lambda v$ (idea: solve for λ)

$\Rightarrow Av - \lambda v = 0$

$\Rightarrow Av - \lambda I v = 0$

$\Rightarrow (A - \lambda I) v = 0$

If not, then $(A - \lambda I)^{-1}$ exists, then $v = (A - \lambda I)^{-1} 0 = 0$ which contradicts $v \neq 0$

Since $v \neq 0$, we want $\boxed{\det(A - \lambda I) = 0}$

So our method is to solve this

(First, $\det(A - \lambda I)$ is called the characteristic polynomial of A.
Second $\det(A - \lambda I)$ is a polynomial over λ !!)

Ex Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

Then $A - \lambda I = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$$= \begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}$$

Next $\det(A - \lambda I) = 0$

$$\Rightarrow \det \begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) - 2\cdot 4 = 0$$

$$\Rightarrow 3 - 4\lambda + \lambda^2 - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

$$\Rightarrow (\lambda - 5)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = 5 \text{ OR } \lambda = -1$$

HA!

Lemma λ is an eigenvalue of $A \iff \det(A - \lambda I) = 0$

Pf Next time