Defn $\det(A-\lambda I)$ is called the characteristic polynomial of A

Lemma λ is an eigenvalue of $A \iff \det(A-\lambda I)=0$

pf $(\Rightarrow)$ Last Time

$(\Leftarrow)$ If $\det(A-\lambda I)=0$

then $(A-\lambda I)v=0$ has ∞-many solution

Thus, there is a non-zero vector v so that

$$(A-\lambda I)v=0$$
$$Av-\lambda I v=0$$
$$Av-\lambda v=0$$
$$Av=\lambda v \quad \square$$

Notice that the set of eigenvectors of A that correspond to a particular eigenvalue λ is precisely the set of non-zero solutions to $(A-\lambda I)v=0$

Thus if λ is an eigenvalue of A, we call the null space of $A-\lambda I$ the eigen space corresponding to λ.

So let's find the eigenspaces in our previous example:

$$A=\begin{pmatrix}1&2\\4&3\end{pmatrix}$$

$\lambda=5$

$$A-\lambda I=\begin{pmatrix}1&2\\4&3\end{pmatrix}-\begin{pmatrix}5&0\\0&5\end{pmatrix}=\begin{pmatrix}-4&2\\4&-2\end{pmatrix}$$

Find Null Space

$$\left(\begin{array}{cc|c}-4&2&0\\4&-2&0\end{array}\right) \quad R_2+R_1 \to R_2$$

$$\left(\begin{array}{cc|c}-4&2&0\\0&0&0\end{array}\right)$$

Back Solve

Let $x_2=t$

$-4x_1+2t=0$

$x_1=\frac{1}{2}t$

eigenspace for $\lambda=5$:

$\{t\cdot(\frac{1}{2},1) \mid t\in\mathbb{R}\}$

$\lambda=-1$

$$A-\lambda I=\begin{pmatrix}2&2\\4&4\end{pmatrix}$$

$$\left(\begin{array}{cc|c}2&2&0\\4&4&0\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1&1&0\\0&0&0\end{array}\right)$$

Back solve

Let $x_2=t$

Then $x_1=-t$

eigenspace for $\lambda=-1$

$\{t\cdot(1,-1) \mid t\in\mathbb{R}\}$

Examples

① $A = \begin{pmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{pmatrix}$. Then $\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & -1 & -1 \\ -12 & -\lambda & 5 \\ 4 & -2 & -1-\lambda \end{pmatrix}$

$$= \det\begin{pmatrix} -12 & 5 \\ 4 & -1-\lambda \end{pmatrix} - \lambda \det\begin{pmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{pmatrix} + 2\det\begin{pmatrix} 3-\lambda & -1 \\ -12 & 5 \end{pmatrix}$$

$$= 12+12\lambda-20 - \lambda\left[(3-\lambda)(-1-\lambda)+4\right] + 2\left[15-5\lambda-12\right]$$

$$= 12\lambda-8 - \lambda\left[\lambda^2-2\lambda+1\right] + 6-10\lambda$$

$$= -\lambda^3+2\lambda^2+\lambda-2$$

$$= -(\lambda+1)(\lambda-1)(\lambda-2)$$

Eigenvalues $1, -1, 2$

$\lambda = 1$

$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -12 & -1 & 5 & 0 \\ 4 & -2 & -2 & 0 \end{array}\right) \quad R_3-2R_1$$

$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -12 & -1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_2+6R_1$$

$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & -7 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Back solve $x_3 = t$

$x_2 = -\frac{1}{7}t$

$2x_1 + \frac{1}{7}t - t = 0$

$x_1 = \frac{3}{7}t$

eigenspace for $\lambda=1$

$\{t\cdot(\frac{3}{7}, -\frac{1}{7}, 1) \mid t\in\mathbb{R}\}$

$\lambda = -1$

$$\begin{pmatrix} 4 & -1 & -1 \\ -12 & 1 & 5 \\ 4 & -2 & 0 \end{pmatrix} \quad \begin{matrix} R_2+3R_1 \\ R_3-R_1 \end{matrix}$$

$$\begin{pmatrix} 4 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

eigenspace for $\lambda=-1$

$\{t\cdot(\frac{1}{2}, 1, 1) \mid t\in\mathbb{R}\}$

$\lambda = 2$

$$\begin{pmatrix} 1 & -1 & -1 \\ -12 & -2 & 5 \\ 4 & -2 & -3 \end{pmatrix} \quad \begin{matrix} R_2+12R_1 \\ R_3-4R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & -14 & -7 \\ 0 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Back solve $x_3 = t$

$2x_2 + t = 0$

$x_2 = -\frac{t}{2}$

$x_1 + \frac{t}{2} - t = 0$

$x_1 = \frac{t}{2}$

eigenspace for $\lambda = 2$

$\{t\cdot(\frac{1}{2}, -\frac{1}{2}, 1) \mid t\in\mathbb{R}\}$

Example $B = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$ $\quad \det\begin{pmatrix} 1-\lambda & 2 & 1 \\ 1 & -1-\lambda & 1 \\ 2 & 0 & 1-\lambda \end{pmatrix} = \lambda^3-\lambda^2-5\lambda-3$

$$= (\lambda-3)(\lambda+1)^2$$

Eigenvalues are $\lambda=3$ and $\lambda=-1$ (with multiplicity 2)

$\lambda=-1$

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \; R_2 - \tfrac{1}{2}R_1 \sim \begin{pmatrix} 2 & 2 & 1 \\ 0 & -1 & \tfrac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

eigenspace for $\lambda=-1$ is $\{t\cdot(1, -\tfrac{1}{2}, -1) \mid t\in\mathbb{R}\}$

Check

$$Bv = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -\tfrac{1}{2} \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ \tfrac{1}{2} \\ 1 \end{pmatrix} = (-1)\,v$$

$\lambda=3$

$$\begin{pmatrix} -2 & 2 & 1 \\ 1 & -4 & 1 \\ 2 & 0 & -2 \end{pmatrix} \begin{matrix} \\ R_2 + \tfrac{1}{2}R_1 \\ R_3 + R_1 \end{matrix}$$

$$\begin{pmatrix} -2 & 2 & 1 \\ 0 & -3 & \tfrac{3}{2} \\ 0 & 2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

eigenspace for $\lambda=3$ is $\{t\cdot(1, \tfrac{1}{2}, 1) \mid t\in\mathbb{R}\}$

Example $C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

Find eigenvalues

$$\det(C-\lambda I) = \det\begin{pmatrix} 1-\lambda & 1 & 1 & 1 \\ 0 & -\lambda & 1 & 1 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 1 & 2-\lambda \end{pmatrix} = (1-\lambda)(-\lambda)(1-\lambda)(2-\lambda)$$
$$= -\lambda(1-\lambda)^2(2-\lambda)$$

$\lambda=0$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix}$$

$\lambda=1$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} \\ R_2 + R_1 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Back solve

$x_4 = 0$

$x_3 = 0$

$x_2 = t$

$x_1 = -t$

eigenspace for $\lambda = 0$

$\{ t \cdot (-1, 1, 0, 0) \mid t \in \mathbb{R} \}$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_1 - R_2$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Back solve

$x_4 = t_1$

$x_3 = -t_1$

$x_2 = 0$

$x_1 = t_2$

$(x_1, x_2, x_3, x_4) = (t_2, 0, -t_1, t_1)$

eigenspace for $\lambda = 1$

$\{ t_1 (0, 0, -1, 1) + t_2 (1, 0, 0, 0) \mid t_1, t_2 \in \mathbb{R} \}$

eigenspace for $\lambda = 2$

$\{ t \cdot (1, \frac{1}{3}, 0, \frac{2}{3}) \mid t \in \mathbb{R} \}$

Example Suppose λ is an eigenvalue of A with corresponding eigenvector v. What does A^2 do to v?

$A^2 v = A(Av) = A(\lambda v) = \lambda (Av) = \lambda(\lambda v) = \lambda^2 v$

Claim If A^{-1} exists, then $A^{-1} v = (\frac{1}{\lambda}) v$.

proof $v = (A^{-1}A)v = A^{-1}(Av) = A^{-1}(\lambda v) = \lambda (A^{-1} v)$

$v = \lambda (A^{-1} v)$

$\frac{1}{\lambda} v = A^{-1} v$ $\square$

Example Any triangular matrix

e.g. $A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & 4 & 8 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & \pi \end{pmatrix}$ Then $\det(A - \lambda I) = (1-\lambda)(3-\lambda)(2-\lambda)(\pi - \lambda)$

So the eigenvalues are the diagonal entries!

Example Any diagonal matrix

e.g. If $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ Then the eigenvalues are $1, 2, 3$

Find eigenspaces $\lambda = 1$ $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

eigenspace $\{ t_1(1,0,0,0) + t_2(0,1,0,0) \mid t_1, t_2 \in \mathbb{R} \}$

$\lambda = 2$ $\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}$

eigenspace $\{ t \cdot (0,0,1,0) \mid t \in \mathbb{R} \}$

$\lambda = 3$ $\begin{pmatrix} -2 & & & \\ & -2 & & \\ & & -1 & \\ & & & 0 \end{pmatrix}$

eigenspace $\{ t \cdot (0,0,0,1) \mid t \in \mathbb{R} \}$

So each vector in the standard basis for $\mathbb{R}^4$ is an eigenvector.

Diagonalization

Target Theorem Suppose A is an $n \times n$ matrix. Then A is diagonalizable iff A has n linearly independent eigenvectors.

Defn Suppose A is an $n \times n$ Real matrix. We say A is diagonalizable if we can write $P^{-1}AP = D$ where P, D are Real matrices, and D is diagonal.