7 Diagonalization

We start with an example

$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ $\quad \lambda_1 = -1 \quad v_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad Av_1 = \lambda_1 v_1$

$\lambda_2 = 5 \quad v_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad Av_2 = \lambda_2 v_2$

Since $\{v_1, v_2\}$ forms a basis for $\mathbb{R}^2$, then any vector $u \in \mathbb{R}^2$ can be written as $(*)\ u = c_1 v_1 + c_2 v_2$ for $c_1, c_2 \in \mathbb{R}$

Then $Au = A(c_1 v_1 + c_2 v_2) = Ac_1 v_1 + Ac_2 v_2 = c_1 Av_1 + c_2 Av_2$

$= c_1(-v_1) + c_2(5v_2)$

$(**) = -c_1 v_1 + 5c_2 v_2$

Then if we write $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ and $u = \begin{pmatrix} x \\ y \end{pmatrix}$

Then $(*)\ u = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

And $(**)\ Au = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -c_1 \\ 5c_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}}_{D} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\Rightarrow u = Pc$ and $Au = PDc$

$\Rightarrow Au = APc$

$\Rightarrow APc = PDc$

$\Rightarrow APc - PDc = 0$

$\Rightarrow (AP - PD)c = 0$

$\Rightarrow$ Since this is true for every c,

$AP - PD = 0$

$\Rightarrow AP = PD$

$\Rightarrow P^{-1}AP = D$

(P^{-1} exists, since the columns of P are lin. ind.)

[If $Ax = 0$ for all x, then $A = 0$ (exercise)]

In general

Theorem If A is an $n \times n$ matrix with eigenvalues $\lambda_1 \dots \lambda_n \in \mathbb{R}$ (not necessarily distinct) with corresponding eigenvectors $v_1 \dots v_n \in \mathbb{R}^n$; and if $\{v_1, \dots, v_n\}$ is lin ind, then $P^{-1}AP = D$ where

$$P = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Proof Since $\{v_1, \dots, v_n\}$ is lin ind, every $u \in \mathbb{R}^n$ can be written uniquely as

$u = c_1 v_1 + \dots + c_n v_n$ for some $c_1 \dots c_n \in \mathbb{R}$

Also $Au = A(c_1 v_1 + \dots + c_n v_n) = c_1 A v_1 + \dots + c_n A v_n$
$= \lambda_1 c_1 v_1 + \dots + \lambda_n c_n v_n$

Let $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

Then $u = Pc$ and $Au = P \begin{pmatrix} \lambda_1 c_1 \\ \vdots \\ \lambda_n c_n \end{pmatrix} = PDc$

$Au = APc$

$APc \stackrel{?}{=} PDc$

$\Rightarrow (AP - PD)c = 0$ for all $c \in \mathbb{R}^n$

$\Rightarrow AP - PD = 0$

$\Rightarrow AP = PD$

$\Rightarrow P^{-1}AP = D$ where P^{-1} exists b/c the columns of P are lin. ind. $\square$

Examples $A = \begin{pmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{pmatrix}$ $\quad \lambda_1 = 2 \quad v_1 = (1, -1, 2)$
$\lambda_2 = -1 \quad v_2 = (1, 2, 2)$
$\lambda_3 = 1 \quad v_3 = (1, -\frac{1}{3}, \frac{7}{3})$

Then $P^{-1}AP = D$ where $P = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -\frac{1}{3} \\ 2 & 2 & \frac{7}{3} \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(check this by computing $AP = PD$)

$B = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$ $\quad \lambda_1 = 3 \quad v_1 = (1, \frac{1}{2}, 1)$
$\left.\begin{matrix}\lambda_2 = -1 \\ \lambda_3 = -1\end{matrix}\right\} \rightarrow v_2 = (1, -\frac{1}{2}, -1)$

This is not diagonalizable.

$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ $\quad \lambda_1 = 0 \quad v_1 = (-1, -1, 0, 0)$
$\lambda_2 = 1 \quad v_2 = (1, 0, 0, 0)$
$\lambda_3 = 1 \quad v_3 = (0, 0, 1, -1)$
$\lambda_4 = 2 \quad v_4 = (1, \frac{1}{3}, 0, \frac{2}{3})$

So $P^{-1}CP = D$, where $P = \begin{pmatrix} -1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2/3 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Theorem Suppose A is an $n \times n$ diagonalizable real matrix. Then A has n linearly independent eigenvectors in $\mathbb{R}^n$

Proof Suppose $P^{-1}AP = D$ where A, P, D are real $n \times n$ matrices, and D is a diagonal matrix.

Since P is invertible, its columns are lin ind.

We will show that each column is an eigenvector.

Write $P = (v_1 \dots v_n)$ and $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} = \begin{pmatrix} ae & bf \\ ce & df \end{pmatrix}$

Then $P^{-1}AP = D$

$\Rightarrow AP = PD$

Then $AP = A(v_1 \dots v_n) = (Av_1 \dots Av_n)$

AND $PD = (v_1 \dots v_n)\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = (\lambda_1 v_1 \dots \lambda_n v_n)$

But since $AP = PD$, $Av_1 = \lambda v_1 \dots Av_n = \lambda_n v_n$

Thus $v_1 \dots v_n$ are eigen vectors ! $\square$

(Bonus: $\lambda_1 \dots \lambda_n$ are the eigenvalues of A)

Theorem Suppose A is an $n \times n$ Real matrix and that A has n distinct real eigen values. Then the corresponding eigen vectors are lin ind (and so A is diagonalizable).

Proof Let $\lambda_1, \dots, \lambda_n$ be the (distinct) eigenvalues of A, and let $v_1 \dots v_n$ be corresponding eigen vectors.

Assume toward a contradiction that $\{v_1 \dots v_n\}$ is linearly dependent.

Pick the smallest subset of these $(v_1 \dots v_k)$ that is linearly dependent.

Then $\boxed{c_1 v_1 + \dots + c_k v_k = 0}$ where at least one c_i is not 0.

WLOG ~~$c_i = c_1$~~. $c_i = c_k$

Then $A(c_1 v_1 + \dots + c_k v_k) = A0$

$c_1 Av_1 + \dots + c_k Av_k = 0$

Multiply by λ_1

$$c_1 \lambda_1 v_1 + \dots + c_k \lambda_k v_k = 0$$

$$\lambda_1 c_1 v_1 + \dots + \lambda_k c_k v_k = 0$$

$$\lambda_1 c_1 v_1 + \dots + \lambda_1 c_k v_k = 0$$

Subtract!

$$\underbrace{(\lambda_2 - \lambda_1)} c_2 v_2 + \underbrace{(\lambda_3 - \lambda_1)} c_3 v_3 + \dots + \underbrace{(\lambda_k - \lambda_1)} c_k v_k = 0$$

All non zero, since λ's are distinct

Also $c_k \neq 0$

Thus $\{v_2 \dots v_k\}$ is lin dep $\lightning$ $\square$

Later

Lemma If $P^{-1}AP = B$, then A, B have the same characteristic polynomial

Thm Geometric multiplicity $\leq$ algebraic multiplicity.

An application Let $A = \begin{pmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{pmatrix}$. Then find A^{50}.

Remember that $P^{-1}AP = D$ where $P = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -\frac{1}{3} \\ 2 & 2 & 7/3 \end{pmatrix}$ $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Then $A = PDP^{-1}$

So $A^{50} = \underbrace{(PDP^{-1})(PDP^{-1}) \dots (PDP^{-1})}_{50 \text{ times}} = P D^{50} P^{-1}$

$$= P \begin{pmatrix} 2^{50} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1}$$

!!

Cayley Hamilton Theorem Suppose $P(x)$ is the characteristic polynomial of A. Then $P(A) = 0$.

Exercise Prove CH for 2×2 matrices.

Lemma If $x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ is the characteristic polynomial of A.

Then $a_0 = 0$ iff 0 is an eigenvalue
iff $Ax = 0$ has a non trivial solution
iff A is not invertible.

Lemma Let $P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ be the char poly of A.

Then $a_0 = P(0) = \det(A - 0I) = \det A$

Corollary of CH Suppose A is invertible and $P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ is the char poly of A.

Then $A^{-1} = -\frac{1}{a_0}\left(a_n A^{n-1} + a_{n-1} A^{n-2} + \dots + a_1 I\right)$

pf By CH, $P(A) = 0$

So $a_n A^n + a_{n-1} A^{n-1} + \dots a_2 A^2 + a_1 A + a_0 I = 0$

Then $I = \frac{-1}{a_0}\left(a_n A^n + a_{n-1} A^{n-1} + \dots + a_1 A\right)$

($a_0 \neq 0$, since A is invertible)

$I = \frac{-1}{a_0} A\left(a_n A^{n-1} + a_{n-1} A^{n-2} + \dots + a_1 I\right)$

Thus

$A^{-1} = -\frac{1}{a_0}\left(a_n A^{n-1} + a_{n-1} A^{n-2} + \dots + a_1 I\right)$ $\square$