If $\binom{x}{y}$ is a vector in $\mathbb{R}^2$, then the set of vectors perpendicular to $\binom{x}{y}$ is $\left\{ k\binom{-y}{x} \mid k \in \mathbb{R} \right\}$

(a) How to prove a set of vectors is lin ind?

Assume $c_1 v_1 + c_2 v_2 + \cdots + c_r v_r = 0$

Then prove $c_1 = 0, c_2 = 0, \ldots, c_r = 0$

Ex Prove $\left\{ \binom{1}{2}, \binom{1}{3} \right\}$ is lin ind.

Ans Suppose $c_1\binom{1}{2} + c_2\binom{1}{3} = 0$

$\Rightarrow$ $c_1 + c_2 = 0$

$2c_1 + 3c_2 = 0$

$\Rightarrow \ldots \Rightarrow c_1 = 0, c_2 = 0$ $\square$

(b) How to prove a set of vectors is lin dep?

Find a non trivial lin combination that equals zero

Ex Prove $\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\3\\4\end{pmatrix} \right\}$ is lin dep.

Ans

$$\begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}2\\3\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad \square$$

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## Chapter 7 (continued)

**Cayley Hamilton Theorem** If $p(x)$ $= \det(A - xI)$ is the characteristic polynomial of a matrix $A$, then $p(A) = O$.

proof Consider $\operatorname{adj}(A - xI)$.

Each entry of $\operatorname{adj}(A - xI)$, is a determinant of an $(n-1)\times(n-1)$ submatrix of $(A - xI)$

Thus each entry is a polynomial of $x$ of degree $\leq n-1$

Suppose $A - xI = \begin{pmatrix} 2-x & 3 & 9 \\ 4 & 5-x & 8 \\ 6 & 7 & 10-x \end{pmatrix}$

Thus $\operatorname{adj}(A - xI)$ can be expressed as

$$\text{adj}(A - xI) = x^{n-1}B_{n-1} + x^{n-2}B_{n-2} + \dots + xB_1 + B_0$$

Idea

$$\begin{pmatrix} x^2 & x^2+x+1 \\ 2x+1 & 0 \end{pmatrix} = x^2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + x\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Next let

$$\det(A - xI) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$

Now Recall that $(A - xI)\underbrace{\left[\frac{\text{adj}(A-xI)}{\det(A-xI)}\right]}_{(A-xI)^{-1}} = I$

$$(A - xI)\,\text{adj}(A - xI) = \det(A - xI)\,I$$

$$\Rightarrow (A - xI)\left[x^{n-1}B_{n-1} + x^{n-2}B_{n-2} + \dots + xB_1 + B_0\right] = \left(a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0\right)I$$

Then, matching coefficients for powers of $x$:

$$\begin{aligned}
x^n &: A^n\big(-B_{n-1} = a_n I\big) \\
x^{n-1} &: A^{n-1}\big(AB_{n-1} - B_{n-2} = a_{n-1} I\big) \\
x^{n-2} &: A^{n-2}\big(AB_{n-2} - B_{n-3} = a_{n-2} I\big) \\
&\ \vdots \\
x^1 &: A\big(AB_1 - B_0 = a_1 I\big) \\
x^0 &: \quad AB_0 = a_0 I
\end{aligned}$$

Then adding all of these together, the RHS is

$$a_n A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \dots + a_1 A + a_0 I = p(A)$$

and the LHS is

$$-A^n B_{n-1} + \left(A^n B_{n-1} - A^{n-1}B_{n-2}\right) + \left(A^{n-1}B_{n-2} - A^{n-2}B_{n-3}\right) + \dots + \left(A^2 B_1 - AB_0\right) + AB_0 = 0$$

$\square$

An example with complex eigenvalues:

Let $A = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}$

Then $\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{pmatrix} = (1-\lambda)(-1-\lambda) + 5 = \lambda^2 + 4$

So the eigen values are $\pm 2i$.

Find eigenvectors

$\lambda = 2i$ $\begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix}$ $R_1 - (1-2i)R_2$

$\begin{pmatrix} 0 & 0 \\ 1 & -1-2i \end{pmatrix}$

Back solve $x_2 = t$

$x_1 = (1+2i)t$

eigenspace $\{ t(1+2i, 1) \mid t \in \mathbb{R} \}$

etc (A is diagonalizable over $\mathbb{C}$)

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# Ch 8 Linear Transformations

In this course, "transformation" is another word for "function."

In this chapter, we will study functions from vector spaces to vector spaces.

Defn ① A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if there is an $m \times n$ real matrix $A$ so that

$$T(x) = y \iff Ax = y$$

② In this case, the matrix $A$ is called the standard matrix of $T$.

Example $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Definition A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is called a linear operator if $n = m$.

Examples The zero matrix is the standard matrix for $T(x) = 0$

The Identity matrix is the standard matrix for $T(x) = x$

Lemma Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\langle e_1, \dots, e_n \rangle$ be the standard basis for $\mathbb{R}^n$.

Then the standard matrix for $T$ is

$$A = \begin{pmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{pmatrix}$$

(where each $T(e_i)$ is a column matrix)

**proof** Suppose $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

Then $T(e_i) = Ae_i = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $i^{th}$ position

$= i^{th}$ column of $A$ $\square$

## Interlude Linear Operators in $\mathbb{R}^2$

Using the lemma, if $T$ is the reflection across the x-axis,

then $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

and $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$

thus $A = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

What if $T$ is a dilation by $k$ in the x-coordinate?

$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}k\\0\end{pmatrix}$

$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

So $A = \begin{pmatrix}k & 0\\0 & 1\end{pmatrix}$

~~so~~ + all of our other examples from before...

**Elementary matrices?**

| $A$ | Geometric Result of $A$ |
| --- | --- |
| $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ | Reflection across $x=y$ |
| $\begin{pmatrix}k&0\\0&1\end{pmatrix}$, $\begin{pmatrix}1&0\\0&k\end{pmatrix}$ | dilations |
| $\begin{pmatrix}1&k\\0&1\end{pmatrix}$, $\begin{pmatrix}1&0\\k&1\end{pmatrix}$ | shears |

Since every invertible matrix is the finite product of elem matrices, then

**Prop** Every invertible linear operator in $\mathbb{R}^2$ is the product of shears, dilations + reflections across $X=y$.

End of $\mathbb{R}^2$ interlude //

**Lemma** If $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^p$ are linear transformations, then so is $T_2 \circ T_1$.

**Proof** Suppose $A_1$ is the standard matrix for $T_1$, and that $A_2$ is the standard matrix for $T_2$.

Then $(T_2 \circ T_1)(x) = T_2(T_1(x)) = T_2(A_1 x) = A_2(A_1 x) = (A_2 A_1)x$

Thus $A_2 A_1$ is the standard matrix for $T_2 \circ T_1$ $\square$

Definition 1 A function $f: X \to Y$ is one-to-one (or injective) if $(\forall x_1, x_2 \in X)\ (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$

2 A function $f: X \to Y$ is onto (or surjective) if for all $y \in Y$, there exists $x \in X$ so that $f(x) = y$

2a Equivalently, $f: X \to Y$ is onto if $\text{Range}(f) = Y$.

Proposition Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then TFAE:

1. The standard matrix of $T$ is invertible
2. $T$ is one-to-one.
3. $T$ is onto.