Negative Polarity Items (NPI)

Proposition Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator, with standard matrix A.

TFAE
1. A is invertible
2. T is one-to-one
3. T is onto

Proof $(1 \Rightarrow 2)$ Suppose A is invertible and $T(x_1) = T(x_2)$

$$\Rightarrow Ax_1 = Ax_2$$
$$\Rightarrow A^{-1}(Ax_1) = A^{-1}(Ax_2)$$
$$\Rightarrow (A^{-1}A)x_1 = (A^{-1}A)x_2$$
$$\Rightarrow Ix_1 = Ix_2$$
$$\Rightarrow x_1 = x_2$$

$(2 \Rightarrow 1)$ Since T is one-to-one $T(x) = 0$ has only the solution $x = 0$

$\Rightarrow Ax = 0$ only has the solution $x = 0$

$\Rightarrow A$ is invertible.

$(1 \Rightarrow 3)$ Suppose A is invertible.

Let $x \in \mathbb{R}^n$.

Then $T(A^{-1}x) = AA^{-1}x = x$

$(3 \Rightarrow 1)$ Suppose T is onto. Then for each $1 \leq i \leq n$,

choose a vector $c_i \in \mathbb{R}^n$ so that $T(c_i) = e_i$

Then let $C = (c_1\ c_2 \dots c_n)$

then $AC = A(c_1\ c_2 \dots c_n) = (e_1\ e_2 \dots e_n) = I$

Thus $C = A^{-1}$ $\square$

Fact If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator with standard matrix A, and if A is invertible, then T^{-1} exists, and it has standard matrix A^{-1}.

proof $(T^{-1} \circ T)(x) = A^{-1}Ax = x$

and $(T \circ T^{-1})(x) = AA^{-1}x = x$ $\square$

Theorem A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear iff both of the following

1. $T(u+v) = T(u) + T(v)$

② $T(cu) = cT(u)$ (where c is a scalar)

proof ($\Rightarrow$) Let A be the standard matrix of T.

Then $T(u+v) = A(u+v) = Au + Av = T(u) + T(v)$

And $T(cu) = A(cu) = cAu = cT(u)$

($\Leftarrow$) Assume the 2 properties are true of T. We need to find A.

Let $A = [\,T(e_1)\ \ T(e_2)\ \cdots\ T(e_n)\,]$

Claim $T(x) = Ax$

pf Let $x \in \mathbb{R}^n$. Then we can write $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

So $Ax = \left(T(e_1) \cdots T(e_n)\right)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$= x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n)$

(by property 2) $= T(x_1 e_1) + T(x_2 e_2) + \cdots + T(x_n e_n)$

(by property 1) $= T(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$

$= T(x)$

$\square$

More Abstractly Now

Definition If V and W are real vector spaces, then we say that $T: V \to W$ is linear if

① $T(u+v) = T(u) + T(v)$

and ② $T(cu) = cT(u)$ for any $c \in \mathbb{R}$.

Examples ⓐ $T: V \to V$ defined by $T(x) = x$.] Identity Transformation.

ⓑ $T: V \to W$ defined by $T(x) = 0$

ⓒ Let P be the vector space of all polynomials.

Define $T: P \to P$ by $T(f) = xf$.

pf ⓐ $T(f+g) = x(f+g) = xf + xg = T(f) + T(g)$

ⓑ $T(cf) = xcf = cxf = cT(f)$.

ⓓ Define $T: P \to P$ by $T(f) = f'$

pf ⓐ $T(f+g) = (f+g)' = f' + g' = T(f) + T(g)$

ⓑ $T(cf) = (cf)' = cf' = cT(f)$

ⓔ Define $T: P \to \mathbb{R}$ by $T(f) = \int_0^1 f(x)\,dx$

pf (a) $T(f+g) = \int_0^1 (f+g)(x)\,dx = \int_0^1 (f(x)+g(x))dx = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx$
$= T(f) + T(g)$

(b) etc...

Prop Suppose $T: V \to W$ is a linear transformation, that V is finite dimensional, and that $\{v_1, \ldots v_n\}$ is a basis for V. Then T is determined by $T(v_1), \ldots T(v_n)$.

proof Let $x \in V$.
Then x can be uniquely expressed as $x = c_1 v_1 + \ldots + c_n v_n$

Then
$$T(x) = T(c_1 v_1 + \ldots + c_n v_n)$$
$$= T(c_1 v_1) + \ldots + T(c_n v_n)$$
$$= c_1 T(v_1) + \ldots + c_n T(v_n) \quad \square$$

Proposition If $T: V \to W$ and $S: W \to U$ are linear transformations, then so is ~~ToS~~. $S \circ T$

proof (a) Let $v_1, v_2 \in V$. Then
$$(S\circ T)(v_1+v_2) = S(T(v_1+v_2)) = S(T(v_1)+T(v_2))$$
$$= S(T(v_1)) + S(T(v_2))$$
$$= (S\circ T)(v_1) + (S\circ T)(v_2)$$

(b) Let $v \in V$ and let $c \in \mathbb{R}$. Then
$$(S\circ T)(cv) = S(T(cv)) = S(c\,T(v)) = c\,S(T(v)) = c(S\circ T)(v) \quad \square$$

Change of Basis

Suppose V is a real vector space with basis $\mathcal{B} = \{u_1, \ldots u_n\}$
Then every vector $u \in V$ can be uniquely expressed as $u = \beta_1 u_1 + \beta_2 u_2 + \ldots + \beta_n u_n$
(Thus there is a one-to-one correspondence between vectors $u \in V$ and vectors $(\beta_1, \ldots \beta_n) \in \mathbb{R}^n$)

In this case we call
$$[u]_{\mathcal{B}} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

the coordinate vector of u Relative to $\mathcal{B}$.

The question for this section is how to move from one basis to another.

Ex Let $u_1 = (1,2,1,0)$
$u_2 = (3,3,3,0)$

$u_3 = (2, -10, 0, 0)$
$u_4 = (-2, 1, -6, 2)$

Since $\mathcal{B} = \{u_1, u_2, u_3, u_4\}$ is lin ind, it forms a basis of $\mathbb{R}^4$
Thus any $u = (x, y, z, w) \in \mathbb{R}^4$ can be written uniquely as

$$u = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \beta_1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 3 \\ 3 \\ 3 \\ 0 \end{pmatrix} + \beta_3 \begin{pmatrix} 2 \\ -10 \\ 0 \\ 0 \end{pmatrix} + \beta_4 \begin{pmatrix} -2 \\ 1 \\ -6 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 & -2 \\ 2 & 3 & -10 & 1 \\ 1 & 3 & 0 & -6 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

$$\Rightarrow [u]_{\mathcal{B}} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 & -2 \\ 2 & 3 & -10 & 1 \\ 1 & 3 & 0 & -6 \\ 0 & 0 & 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

This tells us how to find $[u]_{\mathcal{B}}$, but we started with the coordinates of u relative to the standard basis.

What if we have $[u]_{\mathcal{B}}$ and we want to find $[u]_{\mathcal{C}}$?

So suppose $\mathcal{B} = \{u_1, \ldots u_n\}$ and $\mathcal{C} = \{v_1, \ldots v_n\}$ are bases of V.
Then if $u \in V$, u can be uniquely expressed as

$$u = \beta_1 u_1 + \cdots + \beta_n u_n = \gamma_1 v_1 + \cdots + \gamma_n v_n$$

But we can also write each v_i uniquely as a lin comb of $u_1, \ldots u_n$:

Let $v_i = a_{1i} u_1 + a_{2i} u_2 + \cdots + a_{ni} u_n$ (for each $1 \le i \le n$)

Then
$$\begin{aligned} u &= \gamma_1 v_1 + \cdots + \gamma_n v_n \\ &= \gamma_1 (a_{11} u_1 + \cdots + a_{n1} u_n) + \gamma_2 (a_{12} u_1 + \cdots + a_{n2} u_n) + \cdots \\ &\quad + \gamma_n (a_{1n} u_1 + \cdots + a_{nn} u_n) \\ &= (\gamma_1 a_{11} + \gamma_2 a_{12} + \cdots + \gamma_n a_{1n}) u_1 \\ &\quad + (\gamma_1 a_{21} + \gamma_2 a_{22} + \cdots + \gamma_n a_{2n}) u_2 \end{aligned}$$

$$\vdots$$
$$+ (\gamma_1 a_{n1} + \gamma_2 a_{n2} + \dots + \gamma_n a_{nn}) u_n$$

Thus
$$\beta_1 = \gamma_1 a_{11} + \dots + \gamma_n a_{1n}$$
$$\beta_2 = \gamma_1 a_{21} + \dots + \gamma_n a_{2n}$$
$$\vdots$$
$$\beta_n = \gamma_1 a_{n1} + \dots + \gamma_n a_{nn}$$

$$\Rightarrow \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}$$

$$[u]_{\mathcal{B}} = P \quad [u]_{\mathcal{C}}$$

Where the columns of P are the elements of $\mathcal{C}$ represented in $\mathcal{B}$:
$$P = \left([v_1]_{\mathcal{B}} \ [v_2]_{\mathcal{B}} \ \cdots \ [v_n]_{\mathcal{B}} \right)$$

P is called the transition matrix from $\mathcal{C}$ to $\mathcal{B}$.