Change of Basis examples

Ex 8.5.2 Consider the following 2 bases of $\mathbb{R}^4$:

$$\mathcal{B}\begin{cases} u_1 = (1,2,1,0) \\ u_2 = (3,3,3,0) \\ u_3 = (2,-10,0,0) \\ u_4 = (-2,1,-6,2) \end{cases} \qquad \mathcal{C}\begin{cases} v_1 = (1,2,1,0) \\ v_2 = (1,-1,1,0) \\ v_3 = (1,0,-1,0) \\ v_4 = (0,0,0,2) \end{cases}$$

Notice that

$$v_1 = u_1$$
$$v_2 = -2u_1 + u_2$$
$$v_3 = 11u_1 - 4u_2 + u_3$$
$$v_4 = -27u_1 + 11u_2 - 2u_3 + u_4$$

Then let $P = \left([v_1]_{\mathcal{B}}\ [v_2]_{\mathcal{B}}\ [v_3]_{\mathcal{B}}\ [v_4]_{\mathcal{B}}\right) = \begin{pmatrix} 1 & -2 & 11 & -27 \\ 0 & 1 & -4 & 11 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Thus $[u]_{\mathcal{B}} = P[u]_{\mathcal{C}}$, by previous theorem.

For example If $[u]_{\mathcal{C}} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$. (Then $u = v_1 - v_2 + v_3 = (1,3,-1,0)$)

Then $[u]_{\mathcal{B}} = P[u]_{\mathcal{C}} = \begin{pmatrix} 1 & -2 & 11 & -27 \\ 0 & 1 & -4 & 11 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 14 \\ -5 \\ 1 \\ 0 \end{pmatrix}$

And in fact $14u_1 - 5u_2 + u_3 = (1,3,-1,0)$

Example 8.5.3 Now let's work in P_2 = polynomials of degree ≤ 2.

$$\mathcal{B}\begin{cases} u_1 = 1+x \\ u_2 = 1+x^2 \\ u_3 = x+x^2 \end{cases} \qquad \mathcal{C}\begin{cases} v_1 = 1 \\ v_2 = 1+x \\ v_3 = 1+x+x^2 \end{cases}$$

Notice that $v_1 = \dfrac{u_1+u_2-u_3}{2} = \frac{1}{2}u_1 + \frac{1}{2}u_2 - \frac{1}{2}u_3$

$v_2 = u_1$

$v_3 = \dfrac{u_1+u_2+u_3}{2} = \frac{1}{2}u_1 + \frac{1}{2}u_2 + \frac{1}{2}u_3$

Thus $P = \left([v_1]_{\mathcal{B}}\ [v_2]_{\mathcal{B}}\ [v_3]_{\mathcal{B}}\right) = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

Also notice that $u_1 = v_2$

$u_2 = v_3 - v_2 + v_1 = v_1 - v_2 + v_3$

$u_3 = -v_1 + v_3$

So $P^{-1} = \left([u_1]_{\mathcal{E}}\ [u_2]_{\mathcal{E}}\ [u_3]_{\mathcal{E}}\right) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

Then, for example, let $u = -2u_1 + u_2 + 3u_3 = -1 + x + 4x^2$

Then $[u]_{\mathcal{E}} = P^{-1}[u]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}$

check $-2v_1 - 3v_2 + 4v_2 = -2 - 3(1+x) + 4(1+x+x^2)$

$= -1 + x + 4x^2$ ✓

Kernel + Range

Definition If $T: V \to W$ is a linear transformation, then

① $\{v \in V : T(v) = 0\}$ is called the kernel of T.

② $\{w \in W :$ there is some $v \in V$ so that $T(v) = w\}$ is called the Range of T.

Note If A is the standard matrix of T, then

① kernel of T = Null space of A

② Range of T = column space of A

Notation

$\ker(T)$, $\mathrm{Ran}(T)$

Theorem If $T: V \to W$ is a linear transformation, then

① kernel of T is a linear subspace of V

and ② Range of T is a linear subspace of W.

Proof ① Suppose $u, v \in \ker(T)$. Then $T(u)=0$ and $T(v)=0$

So $T(u+v) = T(u) + T(v) = 0+0 = 0$. Hence $u+v \in \ker(T)$

Next suppose $u \in \ker(T)$ and $\alpha \in \mathbb{R}$. Then $T(u)=0$

So $T(\alpha u) = \alpha T(u) = \alpha \cdot 0 = 0$. Hence $\alpha u \in \ker(T)$

② Assume $w_1, w_2 \in Ran(T)$. Then there are $v_1, v_2 \in V$ s.t. $T(v_1) = w_1$ and $T(v_2) = w_2$

Then $T(v_1+v_2) = T(v_1) + T(v_2) = w_1 + w_2$

Hence $w_1 + w_2 \in Ran(T)$

Assume $w \in Ran(T)$ and $\alpha \in \mathbb{R}$. Then there is a $v \in V$ s.t. $T(v) = w$

So $T(\alpha v) = \alpha T(v) = \alpha w$. Hence $\alpha w \in Ran(T)$ $\square$

Examples Ⓐ Let $Id: V \to V$ be defined by $Id(x) = x$

$\ker(Id) = \{0\}$

$ran(Id) = V$

The zero transformation

Ⓑ Let $f: V \to W$ be defined by $f(x) = 0$ for all $x \in V$

$\ker(f) = V$

$ran(f) = \{0\}$

Ⓒ Define $f: \mathbb{R}^3 \to \mathbb{R}^3$ as the orthogonal projection onto the x_1x_2-plane.

$\ker(f) =$ the x_3-axis

$ran(f) =$ the x_1x_2-plane

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Ⓓ Define $f: P_{20} \to P_{20}$ by $f(p) = p'$ (the derivative)

$\ker(f) = P_0$ = set of constant functions $= \mathbb{R}$

$ran(f) = P_{19}$

Ⓔ Define $f: P_{20} \to \mathbb{R}$ by $f(p) = \int_0^1 p(x)\,dx$,

$\ker(f) =$ set of polynomials with $\int_0^1 p(x)\,dx = 0$

$ran(f) = \mathbb{R}$

Rank-Nullity Theorem Let $T: V \to W$ be a linear transformation. Suppose $\dim(V) = n$. Then

$$\underbrace{\dim(\ker(T))}_{\text{Nullity}} + \underbrace{\dim(Ran(T))}_{\text{Rank}} = n$$

(NB we've already proven this for matrix transformations)

<u>Proof</u> <u>Case I</u> $\dim(\ker(T)) = n$

Then $\ker(T) = V$

So $\operatorname{Ran}(T) = \{0\}$

$\Rightarrow \dim(\operatorname{Ran}(T)) = 0$

<u>Case II</u> $\dim(\ker(T)) = 0$

Then $\ker(T) = \{0\}$

<u>Claim</u> $\dim(\operatorname{Ran}(T)) = n$

<u>proof</u> Let $\{v_1, \dots, v_n\}$ be a basis for V.

We will show that $\{T(v_1), \dots, T(v_n)\}$ is linearly ind.

Let $c_1 T(v_1) + \dots + c_n T(v_n) = 0$

linearity: $\Rightarrow T(c_1 v_1) + \dots + T(c_n v_n) = 0$

$\Rightarrow T(c_1 v_1 + \dots + c_n v_n) = 0$

$\ker(T) = 0$: $\Rightarrow c_1 v_1 + \dots + c_n v_n = 0$

$\{v_1, \dots, v_n\}$ is lin. ind.: $\Rightarrow c_1 = 0, c_2 = 0, \dots, c_n = 0$

Thus $\dim(\operatorname{Ran}(T)) \geq n$.

Why is $\dim(\operatorname{ran}(T)) \leq n$? Suppose $w \in \operatorname{ran}(T)$. Then $\exists v \in V$ st $T(v) = w$ and there are scalars $\alpha_1 \dots \alpha_n$ st $v = \alpha_1 v_1 + \dots + \alpha_n v_n$

So $T(\alpha_1 v_1 + \dots + \alpha_n v_n) = w$

$\Rightarrow \alpha_1 T(v_1) + \dots + \alpha_n T(v_n) = w$

So $w \in \operatorname{span}\{T(v_1), \dots, T(v_n)\}$

Hence $\operatorname{Ran}(T) \subseteq \operatorname{span}\{T(v_1) \dots T(v_n)\}$

So $\dim \operatorname{Ran}(T) \leq n$ $\quad \square$ case II

<u>Case III</u> $0 < \dim(\ker(T)) < n$

Let $\dim(\ker(T)) = k$, and let $v_1, \dots, v_k$ be a basis for $\ker(T)$.

Then let $v_1 \dots v_k, v_{k+1}, \dots, v_n$ be a basis for V extending our basis for $\ker(T)$.

<u>Claim</u> $\{T(v_{k+1}), \dots, T(v_n)\}$ is a basis for $\operatorname{Ran}(T)$

<u>proof of claim</u> First let $w \in \operatorname{Ran}(T)$. Then $\exists u \in V$ st $T(u) = w$

Also, there are scalars $\alpha_1, \dots, \alpha_n$ so that

$u = \alpha_1 v_1 + \dots + \alpha_k v_k + \alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n$

So $T(u) = T(\alpha_1 v_1 + \dots + \alpha_k v_k + \alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n)$

$= \alpha_1 \underbrace{T(v_1)}_{0} + \dots + \alpha_k \underbrace{T(v_k)}_{0} + \alpha_{k+1} T(v_{k+1}) + \dots + \alpha_n T(v_n)$

$= \alpha_{k+1} T(v_{k+1}) + \dots + \alpha_n T(v_n)$

IOW, $\operatorname{span}\{T(v_{k+1}), \dots, T(v_n)\} = \operatorname{ran}(T)$

Why is $\{T(v_{k+1}), \dots, T(v_n)\}$ lin ind?

Suppose $\alpha_{k+1} T(v_{k+1}) + \dots + \alpha_n T(v_n) = 0$

$\Rightarrow T(\alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n) = 0$

$\Rightarrow \alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n \in \ker(T)$

$\Rightarrow$ There are scalars $c_1 \dots c_k \in \mathbb{R}$ so that

$\alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n = c_1 v_1 + \dots + c_k v_k$

$\Rightarrow c_1 v_1 + \dots + c_k v_k - \alpha_{k+1} v_{k+1} - \dots - \alpha_n v_n = 0$

$\Rightarrow$ Since $\{v_1, \dots, v_n\}$ is a basis of V,

$c_1 = 0, \dots, c_k = 0, \alpha_{k+1} = 0, \dots, \alpha_n = 0$

Hence $\{T(v_{k+1}), \dots, T(v_n)\}$ is lin ind. $\square$ claim

Therefore $\dim(\operatorname{ran}(T)) = n - r = n - \dim(\ker(T))$ $\square$ case III

$\square$