Ch8 Supplement Change of Basis + Similar Matrices:

Toward proving Geometric Multiplicity $\leq$ Algebraic Multiplicity

Suppose $T: V \to V$ is a linear operator, and that $\mathcal{B}, \mathcal{B}'$ are bases of V. Let A be the standard matrix of T with respect to $\mathcal{B}$
" A' " $\mathcal{B}'$

Suppose $\mathcal{B} = \{v_1, \dots, v_n\}$ and $\mathcal{B}' = \{u_1, \dots, u_n\}$

How are A and A' related?

Well, let $v, w \in V$ so that $T(v) = w$.

Then $A[v]_{\mathcal{B}} = [w]_{\mathcal{B}}$

and $A'[v]_{\mathcal{B}'} = [w]_{\mathcal{B}'}$

So now let $P = \left([u_1]_{\mathcal{B}} \ [u_2]_{\mathcal{B}} \ \cdots \ [u_n]_{\mathcal{B}} \right)$

(then $P^{-1} = \left([v_1]_{\mathcal{B}'} \ [v_2]_{\mathcal{B}'} \ \cdots \ [v_n]_{\mathcal{B}'} \right)$)

The $P[v]_{\mathcal{B}'} = [v]_{\mathcal{B}}$, and $P[w]_{\mathcal{B}'} = [w]_{\mathcal{B}}$

$[w]_{\mathcal{B}'} = P^{-1}[w]_{\mathcal{B}}$

Then
$$\begin{aligned}[w]_{\mathcal{B}'} &= P^{-1}[w]_{\mathcal{B}} \\ &= P^{-1}A[v]_{\mathcal{B}} \\ &= P^{-1}AP[v]_{\mathcal{B}'}\end{aligned}$$

This means that $P^{-1}AP$ represents T with respect to the basis $\mathcal{B}'$

So $\boxed{P^{-1}AP = A'}$

Definition If A and A' are matrices so that there is an invertible matrix P satisfying $\boxed{P^{-1}AP = A'}$ then we say that A and A' are similar.

So we just proved that if A, A' are standard matrices of the same linear operator with respect to different bases, then they are similar.

Lemma Similarity is an equivalence relation.

Proof Reflexive Let A be a matrix

Then $I^{-1} A I = A$, so A is similar to A.

Symmetric Suppose A, A' are similar.

Then $P^{-1} A P = A'$

$\Rightarrow A = P A' P^{-1}$

$\Rightarrow P A' P^{-1} = A$

$\Rightarrow (P^{-1})^{-1} A' P^{-1} = A$

So A' and A are similar

Transitivity Suppose A, B are similar and that B, C are similar.

Then $P^{-1}AP = B$ and $R^{-1} B R = C$

$\Rightarrow R^{-1} P^{-1} A P R = C$

$\Rightarrow (PR)^{-1} A (PR) = C$

so A and C are similar $\square$

Theorem Suppose A and A' are similar matrices then

1. $\text{Rank}(A) = \text{Rank}(A')$
2. $\det(A) = \det(A')$
3. $\chi(A) = \chi(A')$ ($\chi(A)$ = characteristic polynomial of A)
   ($\Rightarrow A, A'$ have the same eigenvalues!)
4. $\text{Trace}(A) = \text{Trace}(A')$

Definition $\text{Trace}(A)$ = sum of diagonal entries of A.

Proof Suppose $P^{-1}AP = A'$.

(2) $\det(A') = \det(P^{-1} A P) = \det(P^{-1}) \det(A) \det(P)$

$$= \frac{1}{\det(P)} \det(A) \det(P)$$

$$= \det(A)$$

(3) $\chi(A') = \det(A' - \lambda I)$

$$= \det(P^{-1}AP - \lambda \boxed{I})$$

$$= \det(P^{-1}AP - \lambda \boxed{P^{-1} I P})$$

$$= \det(P^{-1}AP - P^{-1} \lambda I P)$$

$$= \det(P^{-1} [AP - \lambda I P])$$

$$= \det(P^{-1} [A - \lambda I] P)$$

$= \det(P^{-1}) \det(A-\lambda I) \det(P)$

$= \frac{1}{\det(P)} \det(A-\lambda I) \det(P)$

$= \det(A-\lambda I)$

$= \chi(A)$

(1) **Exercise** $\mathrm{Rank}(AB) \leq \mathrm{Rank}(B)$ **AND** $\mathrm{Rank}(AB) \leq \mathrm{Rank}(A)$

So if $A' = P^{-1}AP$, then $\mathrm{Rank}(A') = \mathrm{Rank}(P^{-1}AP)$

$\leq \mathrm{Rank}(A)$

AND $\mathrm{Rank}(A) = \mathrm{Rank}(PA'P^{-1}) \leq \mathrm{Rank}(A')$

Therefore $\mathrm{Rank}(A) = \mathrm{Rank}(A')$

(4) **Lemma** $\mathrm{Trace}(AB) = \mathrm{Trace}(BA)$

Let's explore this for 2x2 matrices:

$$\mathrm{Trace}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right] = \mathrm{Trace}\left[\begin{pmatrix} ae+bg & x \\ x & cf+dh \end{pmatrix}\right] = ae+bg+cf+dh$$

$$\mathrm{Trace}\left[\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \mathrm{Trace}\left[\begin{pmatrix} ea+fc & x \\ x & gb+hd \end{pmatrix}\right] = ea+fc+gb+hd$$

Proof of Lemma Let (c_{ij}) be the entries of AB

Let (d_{ij}) be the entries of BA

Then $\mathrm{Trace}(AB) = \sum_{k=1}^{n} (c_{kk}) = \sum_{k=1}^{n} \left(\sum_{l=1}^{n} a_{kl} b_{lk} \right)$

$= \sum_{l=1}^{n} \left(\sum_{k=1}^{n} b_{lk} a_{kl} \right) =$

$\sum_{l=1}^{n} d_{ll} = \mathrm{Trace}(BA)$ $\square$ lemma

Back to the Theorem! $\mathrm{Trace}(A') = \mathrm{Trace}((P^{-1}A)P)$

$= \mathrm{Trace}((P(P^{-1}A))$

$= \mathrm{Trace}(A)$ $\square$

(**Theorem** $\mathrm{Trace}(A) =$ sum of eigenvalues of A. pf omitted)

Back to our target theorem for today...

Definition Suppose λ is an eigenvalue of A.

1. The geometric multiplicity of λ is the dimension of the corresponding eigenspace
   (= # of linearly ind eigenvectors of λ)
2. The algebraic multiplicity of λ is the multiplicity of λ in $\chi(A)$
   = # of factors of $(x-\lambda)$ in $\chi(A)$

Example Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $\chi(A) = \det\begin{pmatrix} 1-x & 1 \\ 0 & 1-x \end{pmatrix} = (1-x)^2$

So $\lambda = 1$ has algebraic multiplicity of 2.

Eigenspace for $\lambda=1$: $A - 1I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ Back solve $x_2 = 0$, $x_1 = t$

so $v = \begin{pmatrix} 0 \\ t \end{pmatrix}$

So $\lambda=1$ has geometric multiplicity of 1

Theorem If λ_0 is an eigenvalue for A, then

geometric multiplicity of λ_0 $\leq$ algebraic multiplicity of λ_0

Proof Let A be an $n \times n$ matrix, and let λ_0 be an eigenvalue for A.
Suppose λ_0 has geometric multiplicity k. $(k \leq n)$
Then let $v_1, \dots, v_k$ be a set of linearly independent eigenvectors of A corresponding to λ_0
Now extend $v_1, \dots, v_k$ to a basis for $\mathbb{R}^n$:

$$v_1, \dots, v_k, v_{k+1}, \dots, v_n$$

Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$;
And let T be the linear transformation whose standard matrix with respect to $e_1 \dots e_n$ is A.
Next let A' be the standard matrix for T with respect to $v_1 \dots v_n$.

Then A' looks like:

$$\left(\begin{array}{ccc|c} \lambda_0 & & 0 & \\ & \lambda_0 & & * \\ 0 & & \ddots\ \lambda_0 & \\ \hline & 0 & & * \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} k \\ \left.\vphantom{1}\right\} n-k \end{array}$$

$\underbrace{\qquad\qquad}_{k}\ \underbrace{\qquad}_{n-k}$

(Why? $\lambda_0 v_1 = Av_1 = T(v_1)$, but v_1 is a basis vector for A', so the first column of A' is $\begin{pmatrix}\lambda_0\\0\\0\\\vdots\\0\end{pmatrix}$)

So then $\chi(A')$ has a factor of $(\lambda_0 - x)^k$.

And since A and A' are standard matrices of the same linear operator with respect to different bases, they are similar.

So by our theorem about similar matrices, $\chi(A) = \chi(A')$.

Thus $\chi(A)$ also has a factor of $(\lambda_0 - x)^k$.

Therefore Algebraic multiplicity of λ_0 in $A \geq k$ = geometric multiplicity of λ_0 in A $\square$