Inverse linear transformations

Prop A transformation $T: V \to M$ is one-to-one iff $Ker(T) = \{0\}$.

proof ($\Rightarrow$) Suppose $Ker(T) \neq \{0\}$ Suppose $x \in V$ and $x \neq 0$ and $T(x) = 0$.

Then T is not one-to-one because $T(x) = T(0)$ but $x \neq 0$.

($\Leftarrow$) Now suppose $ker(T) = \{0\}$.

Now to show that T is one-to-one, assume $T(x_1) = T(x_2)$ for some $x_1, x_2 \in V$.

$$
\begin{aligned}
\text{Then} \quad & T(x_1) - T(x_2) = 0 \\
\Rightarrow \quad & T(x_1 - x_2) = 0 \quad \text{by linearity} \\
\Rightarrow \quad & x_1 - x_2 = 0 \quad \text{because } ker(T) = \{0\}. \\
\Rightarrow \quad & x_1 = x_2 \quad \square
\end{aligned}
$$

Prop Let $T: V \to V$ be a linear operator. Then TFAE:

(a) T is one-to-one

(b) $Ker(T) = \{0\}$

(c) $Range(T) = V$

Proof ($a \Leftrightarrow b$) just proved

($b \Leftrightarrow c$) by the rank-nullity theorem $\square$

Chapter 9 Real Inner Product Spaces

Example In $\mathbb{R}^n$, $u \cdot v = u_1 v_1 + \cdots + u_n v_n$

and $\|u\| = (u_1^2 + u_2^2 + \cdots + u_n^2)^{\frac{1}{2}} = (u \cdot u)^{\frac{1}{2}}$

and $d(u,v) = ((u_1 - v_1)^2 + \cdots + (u_n - v_n)^2)^{\frac{1}{2}} = \|u - v\| = ((u-v)\cdot(u-v))^{\frac{1}{2}}$

Recall that the dot product satisfies the following definition:

Defn Suppose V is a real vector space. A function

$$\langle \, , \, \rangle : V \times V \to \mathbb{R}$$

is a real inner product on V if

Symmetry (1) $\langle u, v \rangle = \langle v, u \rangle$

additivity (2) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$

homogeneity ③ $c\langle u,v\rangle = \langle cu, v\rangle$

Positive definite ④ $\langle u,u\rangle \geq 0$ and $\langle u,u\rangle = 0$ iff $u=0$

Then V is called a real inner product space

Motivating Example dot product on $\mathbb{R}^n$.

Defn If V is a real inner product space, and $u,v\in V$, we define
① The Norm of u : $\|u\| = \langle u,u\rangle^{\frac{1}{2}}$
② The distance $d(u,v) = \|u-v\| = \langle u-v, u-v\rangle^{\frac{1}{2}}$

Example ① Define $\langle u,v\rangle$ on $\mathbb{R}^3$ by $\langle u,v\rangle = 5u_1v_1 + 7u_2v_2 + u_3v_3$
(NB: The coefficients ~~must~~ be positive)

Ex ② Let A be any INVERTIBLE $n\times n$ matrix. For $u,v\in\mathbb{R}^n$ define
$$\langle u,v\rangle = (Au)\cdot(Av)$$
proof ① $\langle u,v\rangle = (Au)\cdot(Av) = (Av)\cdot(Au) = \langle v,u\rangle$
② $\langle u, v+w\rangle = (Au)\cdot(A(v+w)) = (Au)\cdot[Av+Aw]$
$= (Au)\cdot(Av) + (Aw)\cdot(Aw)$
$= \langle u,v\rangle + \langle u,w\rangle$
③ $c\langle u,v\rangle = c\,(Au)\cdot(Av) = (cAu)\cdot(Av) = (Acu)\cdot(Av) = \langle cu,v\rangle$
④ Ⓐ $\langle u,u\rangle = (Au)\cdot(Au) \geq 0$ because dot product is pos. def.
Ⓑ $\langle u,u\rangle = (Au)\cdot(Au) = 0$ iff $Au=0$
iff $u=0$ since A is invertible. $\square$

Ex ③ Let V be the space of all 2×2 real matrices.
Define $\langle A,B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$
(essentially, this is just the dot product on $\mathbb{R}^4$, as is the next one)

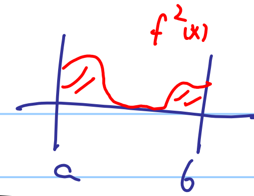
Ex ④ Let P_3 be the vector space of polynomials of degree ≤ 3.
Define $\langle a_0 + a_1x + a_2x^2 + a_3x^3,\ b_0 + b_1x + b_2x^2 + b_3x^3\rangle$
$= a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3$

Ex ⑤ Let V be the vector space of all continuous functions $f:[a,b]\to\mathbb{R}$.

where $[a,b]$ is some closed interval on the real line.

Define $\langle f, g\rangle = \int_a^b (f\cdot g)(x)\,dx$

Then $\|f\| = \int_a^b f^2(x)\,dx$



Recall that for the dot product $u\cdot v = \|u\|\cdot\|v\|\cos\theta$ where θ is the angle between u and v.

But if V is an abstract RIPS, what do we mean by θ?

Nothing, but we can prove that some such θ exists, by proving:

Cauchy-Schwartz Inequality If u and v are vectors in a RIPS, then

$$|\langle u, v\rangle| \le \|u\|\cdot\|v\|$$

This will imply that $-1 \le \dfrac{\langle u,v\rangle}{\|u\|\cdot\|v\|} \le 1$

and so we will call

$$\theta = \cos^{-1}\left(\frac{\langle u,v\rangle}{\|u\|\,\|v\|}\right)$$ the angle between u and v.

And we also still have:

Defn u and v are orthogonal if $\langle u, v\rangle = 0$

Proof of C-S

case I $u = 0$.

Then $\langle u, v\rangle = \langle 0, v\rangle = \langle 0\cdot u, v\rangle \overset{\text{Property 3}}{=} 0\,\langle u, v\rangle = 0$

and $\|u\|\cdot\|v\| = \langle u,u\rangle\cdot\langle v,v\rangle \overset{\text{Property 4}}{\ge} 0$

Case II $u \ne 0$

Then $\langle u, u\rangle > 0$.

Also, if $t\in\mathbb{R}$, then $0 \overset{\text{Prop 4}}{\le} \langle tu+v, tu+v\rangle \overset{\text{Prop 2}}{=} \langle tu+v, tu\rangle + \langle tu+v, v\rangle$

$\overset{P1}{=} \langle tu, tu+v\rangle + \langle v, tu+v\rangle$

$\overset{P2}{=} \langle tu, tu\rangle + \langle tu, v\rangle + \langle v, tu\rangle + \langle v, v\rangle$

$\overset{P3}{=} t\langle u, tu\rangle + t\langle u, v\rangle + \langle v, tu\rangle + \langle v, v\rangle$

$\overset{P1}{=} t\langle tu, u\rangle + t\langle u, v\rangle + \langle tu, v\rangle + \langle v, v\rangle$

$\overset{P3}{=} t^2\langle u,u\rangle + t\langle u,v\rangle + t\langle u,v\rangle + \langle v,v\rangle$

$$= t^2 \underbrace{\boxed{\langle u,u\rangle}}_{\neq 0} + 2t\langle u,v\rangle + \langle v,v\rangle$$

Since this quadratic function of t is not ever negative, it has 1 or 0 real zeros. Thus the discriminant is less than or equal to 0.

$$b^2 - 4ac \leq 0$$

$$[2\langle u,v\rangle]^2 - 4\langle u,u\rangle\langle v,v\rangle \leq 0$$

$$4(\langle u,v\rangle)^2 \leq 4\langle u,u\rangle\langle v,v\rangle$$

$$(\langle u,v\rangle)^2 \leq \langle u,u\rangle\langle v,v\rangle$$

$$(\langle u,v\rangle)^2 \leq \|u\|^2\|v\|^2 \qquad x^2 < 4$$

$$|\langle u,v\rangle| \leq \|u\|\cdot\|v\| \qquad \Rightarrow |x| < 2$$

$\square$

Applications of CSI

① Let $a, b \geq 0$. Let $u = \begin{pmatrix}\sqrt{a}\\ \sqrt{b}\end{pmatrix}$ $v = \begin{pmatrix}\sqrt{b}\\ \sqrt{a}\end{pmatrix}$ be vectors in $\mathbb{R}^2$

Then by CSI,

$$|u\cdot v| \leq \|u\|\cdot\|v\|$$

$$|\sqrt{ab}+\sqrt{ab}| \leq \left(\sqrt{a+b}\right)\left(\sqrt{a+b}\right)$$

$$2\sqrt{ab} \leq a+b$$

$$\sqrt{ab} \leq \frac{a+b}{2}$$

Geometric Mean $\leq$ Arithmetic Mean

② For the dot product in $\mathbb{R}^n$: $u\cdot v \leq \|u\|\cdot\|v\|$

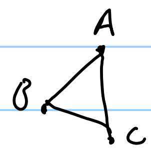
$$\Rightarrow u_1v_1 + u_2v_2 + \dots + u_nv_n \leq (u_1^2 + \dots + u_n^2)^{\frac{1}{2}}(v_1^2 + \dots + v_n^2)^{\frac{1}{2}}$$

③ (Using example 5 above) If $\langle f,g\rangle = \int_a^b (f\cdot g)(x)\,dx$

Then CSI implies

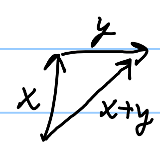
$$\left|\int_a^b (f\cdot g)(x)\,dx\right| \leq \left(\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}}\cdot\left(\int_a^b g^2(x)\,dx\right)^{\frac{1}{2}}$$

④ Triangle Inequality ⓐ For real #'s $|a+b| \leq |a| + |b|$



ⓑ For triangles $d(A,B) + d(B,C) \geq d(A,C)$

ⓒ In terms of vectors:



$\|x+y\| \leq \|x\| + \|y\|$

Proof of ⓒ In a RIPS $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle$

$$= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$$

$$\overset{\text{CSI}}{\leq} \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

Thus $\|x+y\|^2 \leq (\|x\| + \|y\|)^2$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$ $\square$