Orthogonality

Recall that we define u and v to be orthogonal if $\langle u, v\rangle = 0$.

Defn If $W \subseteq V$ is a subspace, we define
$$W^\perp = \{u \in V : \langle u, w\rangle = 0 \text{ for all } w \in W\}$$
the orthogonal complement to W.

Lemma If W is a subspace of V, then so is $W^\perp$.

proof (a) Let $u, v \in W^\perp$.

Then $\langle v, w\rangle = 0$ and $\langle u, w\rangle = 0$ for all $w \in W$

Thus $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle = 0 + 0 = 0$ for all $w \in W$.

So $u + v \in W^\perp$ ← additivity

(b) Suppose $u \in W^\perp$ and $c \in \mathbb{R}$.

Then $\langle u, w\rangle = 0$ for all $w \in W$ ← homogeneity

Thus $\langle cu, w\rangle = c\langle u, w\rangle = c \cdot 0 = 0$ for all $w \in W$.

So $cu \in W^\perp$ □

Examples (1) In $\mathbb{R}^3$, orthogonal $\equiv$ perpendicular

So if W is a plane, then $W^\perp$ is a line, and vice versa.

(2) $\{0\}^\perp = V$ and $V^\perp = \{0\}$

(3) In $\mathbb{R}^4$, let $W = \{(a,b,c,d) \in \mathbb{R}^4 \mid a = 0 \text{ and } c = 0\}$

Then $W^\perp = \{(a,b,c,d) \in \mathbb{R}^4 \mid b = 0 \text{ and } d = 0\}$

proof. Let $A = \{(a,b,c,d) \in \mathbb{R}^4 \mid b = 0 \text{ and } d = 0\}$

$A \subseteq W^\perp$ Suppose $u \in A$. Then $u = (u_1, 0, u_3, 0)$

Then let $w \in W$. Then $w = (0, w_2, 0, w_4)$

So $\langle u, w\rangle = 0$

Thus $u \in W^\perp$

$W^\perp \subseteq A$ Let $u \in W^\perp$. Then for all $w \in W$, $\langle u, w\rangle = 0$.

Write $u = (u_1, u_2, u_3, u_4)$

Pick ~~some~~ $w = (0, \cancel{w_2}\,0, 0, \cancel{w_4}\,1) \in W$

Then $0 = \langle u, w \rangle = \cancel{u_2 w_2 + u_4 w_4}$ u_4

Also, $(0,1,0,0) \in W$. so $0 = \langle u, (0,1,0,0) \rangle = u_2$

Hence $u \in A$ □

<u>Example</u> Let $V = P_2$. Consider $x^2 + 2x + 1 \in P_2$.
Then $x^2 + 2x + 1$ is orthogonal to <u>$x^2 - x + 1$</u>

<u>Example</u> Let A be any $n \times m$ matrix. Suppose $u \in \text{Nullspace}(A)$

$$A \quad u = 0$$
$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \begin{pmatrix} u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Let $v_1, \dots, v_n$ be the rows of A.

Then $v_i \cdot u = 0$ for every $1 \leq i \leq n$

Then if $\sum_{i=1}^{n} c_i v_i$ is any element of the rowspace of A,

Then $u \cdot \left(\sum_{i=1}^{n} c_i v_i \right) = \sum_{i=1}^{n} \left(u \cdot (c_i v_i) \right) = \sum_{i=1}^{n} c_i (u \cdot v_i)$

$$= \sum_{i=1}^{n} c_i \cdot 0 = 0$$

Thus u is orthogonal to every vector in the row space,
so $u \in (\text{Row space}(A))^{\perp}$

Hence $\text{Null space}(A) \subseteq (\text{Rowspace}(A))^{\perp}$!!

<u>Claim</u>: $(\text{Rowspace}(A))^{\perp} \subseteq \text{Nullspace}(A)$

<u>proof</u> Suppose $u \in (\text{Rowspace}(A))^{\perp}$

Thus u is orthogonal to every vector in the row space.

$$\text{So } Au = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \begin{pmatrix} u \end{pmatrix} = \begin{pmatrix} v_1 \cdot u \\ v_2 \cdot u \\ \vdots \\ v_n \cdot u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus $u \in \text{Nullspace}(A)$ □

<u>Conclusion</u> (Thm) $(\text{Rowspace}(A))^{\perp} = \text{Null space}(A)$

[Also, since $\text{Column space}(A^T) = \text{Row space}(A)$
we get $(\text{column space}(A^T))^{\perp} = (\text{Row space}(A))^{\perp} = \text{Null space}(A)$]

<u>Lemma</u> Let V be an n dimensional vector space, and let W be a subspace.

(a) $(W^{\perp})^{\perp} = W$. (b) Thus $W = U^{\perp} \iff U = W^{\perp}$

(c) Also $\dim W^{\perp} = n - \dim W$

<u>proof</u> (c) Let $v_1, \dots, v_k$ be a basis for W.

Construct the $n \times n$ matrix A with $v_1, \dots, v_k, 0, \dots, 0$ as rows.

Then Rowspace$(A) = W$, so $W^\perp = (\text{Rowspace}(A))^\perp = \text{Nullspace}(A)$

So by the Rank-Nullity Theorem:

$$\dim(\text{Rowspace}(A)) + \dim(\text{Nullspace}(A)) = n$$
$$\Rightarrow \dim W + \dim W^\perp = n$$

(a) Next we see that $(W^\perp)^\perp \supseteq W$

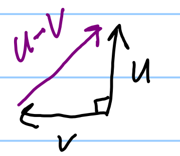
(Why? Let $u \in W$. Then u is orthogonal to every $v \in W^\perp$. Hence $u \in (W^\perp)^\perp$

Next $\dim (W^\perp)^\perp = n - \dim(W^\perp) = n - (n - \dim(W)) = \dim W$

Therefore, since W is a subspace of $(W^\perp)^\perp$ and they have the same dimension, they must be equal.

(b) $W = U^\perp \Rightarrow W^\perp = (U^\perp)^\perp \underset{\text{by part a}}{\Rightarrow} W^\perp = U$ $\square$

Pythagorean Theorem If u and v are orthogonal, then $\|u\|^2 + \|v\|^2 = \|u-v\|^2$



proof Suppose $\langle u, v\rangle = 0$

Then
$$\|u-v\|^2 = \langle u-v, u-v\rangle$$
$$= \langle u, u-v\rangle - \langle v, u-v\rangle$$
$$= \langle u, u\rangle - \underbrace{\langle u, v\rangle - \langle v, u\rangle}_{\text{both } 0} + \langle v, v\rangle$$
$$= \langle u, u\rangle + \langle v, v\rangle$$
$$= \|u\|^2 + \|v\|^2 \quad \square$$

Orthonormal Bases

Definition Suppose $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis for V

(A) If $\langle v_i, v_j\rangle = 0$ for every $1 \le i \le n$ and $1 \le j \le n$ where $i \ne j$ then we say $\{v_1, \ldots, v_n\}$ is pairwise orthogonal

In this case, we say $\mathcal{B}$ is an orthogonal basis.

(B) If $\mathcal{B}$ is an orthogonal basis, and if $\|v_i\| = 1$ for every $1 \le i \le n$, then we say $\mathcal{B}$ is an orthonormal basis

Example In $\mathbb{R}^n$, the standard basis is orthonormal

Example $\{(1,1),(1,-1)\}$ is an orthogonal basis for $\mathbb{R}^2$.

Then $\left\{ \frac{(1,1)}{\sqrt{2}}, \frac{(1,-1)}{\sqrt{2}} \right\}$ is an orthonormal basis for $\mathbb{R}^2$

This is the normalization of this

Proposition Suppose V is a finite-dimension real inner-product space, and that $\{v_1, \ldots, v_n\}$ is an orthogonal basis. Let $u \in V$.

$$\text{Then} \quad u = \frac{\langle u, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle u, v_2 \rangle}{\|v_2\|^2} v_2 + \cdots + \frac{\langle u, v_n \rangle}{\|v_n\|^2} v_n$$

Thus if $\{v_1, \ldots, v_n\}$ is an orthonormal basis,

$$u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 + \cdots + \langle u, v_n \rangle v_n$$

Proof Let $u \in V$. Since $\{v_1, \ldots, v_n\}$ is a basis, we can write

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \quad \text{for some } c_1, \ldots, c_n \in \mathbb{R}.$$

Let $1 \le i \le n$. Then

$$\begin{aligned}
\langle u, v_i \rangle &= \langle c_1 v_1 + c_2 v_2 + \cdots + c_n v_n, v_i \rangle \\
&= c_1 \underbrace{\langle v_1, v_i \rangle}_{=0} + c_2 \underbrace{\langle v_2, v_i \rangle}_{=0} + \cdots + c_i \underbrace{\langle v_i, v_i \rangle}_{\neq 0} + \cdots + c_n \underbrace{\langle v_n, v_i \rangle}_{=0} \\
&= c_i \langle v_i, v_i \rangle \\
&= c_i \|v_i\|^2
\end{aligned}$$

Thus $c_i = \dfrac{\langle u, v_i \rangle}{\|v_i\|^2}$ as desired $\square$