Lemma If $v_1 \ldots v_k$ are pairwise orthogonal (and nonzero) then $\{v_1, \ldots v_k\}$ is linearly independent.

Proof Suppose $v_1 \ldots v_k$ are pairwise orthogonal, and that

$$c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0 \quad \text{for some } c_1 \ldots c_k \in \mathbb{R}$$

We will prove that $c_i = 0 \quad \forall i$:

$$0 = \langle 0, v_i \rangle = \langle c_1 v_1 + c_2 v_2 + \ldots + c_k v_k , v_i \rangle$$

$$= \langle c_1 v_1, v_i \rangle + \langle c_2 v_2, v_i \rangle + \ldots + \langle c_k v_k, v_i \rangle$$

$$= c_1 \langle v_1, v_i \rangle + c_2 \langle v_2, v_i \rangle + \ldots + c_k \langle v_k, v_i \rangle$$

(all of these inner prods are 0, except $\langle v_i, v_i \rangle$)

$$= c_i \langle v_i, v_i \rangle$$

Thus $0 = c_i \langle v_i, v_i \rangle$

and so $c_i = 0$ since $\langle v_i, v_i \rangle \neq 0$ □

Proposition Every finite-dimensional RIPS with dimension ≥ 1 has an orthogonal basis (and thus an orthonormal basis)

Proof (Gram-Schmidt Process)

First let V be a finite-dimensional RIPS with basis $\{u_1, \ldots u_n\}$. We will *construct an orthonormal basis $\{v_1, \ldots v_n\}$. Recursively

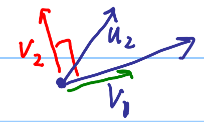
First let $v_1 = u_1$

Next let $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

(we already showed $\langle v_1, v_2 \rangle = 0$)

So what about $v_3, \ldots v_n$?

Idea



$$v_2 = u_2 - \text{proj}_{v_1}(u_2)$$

For inner products, we want to find $v_2 = u_2 - \alpha v_1$ so that $\langle v_2, v_1 \rangle = 0$

$$\langle u_2 - \alpha v_1, v_1 \rangle = 0$$

$$\langle u_2, v_1 \rangle - \alpha \langle v_1, v_1 \rangle = 0$$

$$\Rightarrow \alpha = \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2}$$

Inductive Step

Suppose $v_1 \ldots v_s$ are already pairwise orthogonal. How can modify u_{s+1} to be able to include it as well?

We want $v_{s+1} = u_{s+1} - \alpha_1 v_1 - \alpha_2 v_2 - \dots - \alpha_s v_s$ to be orthogonal to each of the $v_1 \dots v_s$

In other words if $1 \le i \le s$, we want $\langle v_{s+1}, v_i \rangle = 0$

OR $0 = \langle v_i, v_{s+1} \rangle = \langle v_i, u_{s+1} - \alpha_1 v_1 - \alpha_2 v_2 - \dots - \alpha_s v_s \rangle$

$= \langle v_i, u_{s+1} \rangle - \alpha_1 \langle v_i, v_1 \rangle - \alpha_2 \langle v_i, v_2 \rangle \dots - \alpha_s \langle v_i, v_s \rangle$

$= \langle v_i, u_{s+1} \rangle - \alpha_i \langle v_i, v_i \rangle$ (most of these are 0 by IH)

OR. $0 = \langle v_i, u_{s+1} \rangle - \alpha_i \langle v_i, v_i \rangle$

So we set $\alpha_i = \dfrac{\langle v_i, u_{s+1} \rangle}{\langle v_i, v_i \rangle} = \dfrac{\langle v_i, u_{s+1} \rangle}{\|v_i\|^2}$

In conclusion, set

$$v_{s+1} = u_{s+1} - \sum_{i=1}^{s} \frac{\langle v_i, u_{s+1} \rangle}{\|v_i\|^2} v_i$$

Claim For every $1 \le i \le n$, $v_i \ne 0$.

proof Assume toward a contradiction that some $v_i = 0$.

$$u_{s+1} - \sum_{i=1}^{s} \frac{\langle v_i, u_{s+1} \rangle}{\|v_i\|^2} v_i = 0$$

$$u_{s+1} = \sum_{i=1}^{s} \frac{\langle v_i, u_{s+1} \rangle}{\|v_i\|^2} v_i$$

So u_{s+1} is a linear comb of $\{v_1, \dots v_s\}$, so it's a linear comb of $\{u_1, \dots u_s\}$

↯ □ Claim

So thus by the previous prop, since the v_i's are pairwise orthogonal and nonzero, they are also lin ind □

Example Let V be the subspace of $\mathbb{R}^4$ spanned by the vectors

$u_1 = (1,2,1,0)$

$u_2 = (3,3,3,0)$

$u_3 = (2,1,-6,2)$

First note that $\{u_1, u_2, u_3\}$ is lin ind.

So let's use Gram-Schmidt Orthogonalization Process to find an orthogonal basis for V.

First set $v_1 = (1,2,1,0)$

Next let $V_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1 = (3,3,3,0) - \frac{12}{6}(1,2,1,0)$

$= (3,3,3,0) - (2,4,2,0)$

$= (1,-1,1,0)$

Next let $v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$

$= (-2,1,-6,2) - \frac{-6}{6}(1,2,1,0) - \frac{-9}{3}(1,-1,1,0)$

$= (-2,1,-6,2) + (1,2,1,0) + (3,-3,3,0)$

$= (2,0,-2,2)$

So $\{(1,2,1,0), (1,-1,1,0), (2,0,-2,2)\}$ is an orthogonal basis for V

Prop Suppose V is a finite-dimensional RIPS, and that $W \leq V$ is a subspace. If $u \in V$ then there is a unique way to write $u = w + p$ with $w \in W$ and $p \in W^\perp$

namely if $\{v_1, \dots, v_k\}$ is an orthogonal basis for W then

$$w = \frac{\langle u, v_1\rangle}{\|v_1\|^2} v_1 + \dots + \frac{\langle u, v_k\rangle}{\|v_k\|^2} v_k$$

and then $p = u - w$

Proof Let $\{v_1, \dots, v_k\}$ be an orthogonal basis for W.

Extend $\{v_1, \dots, v_k\}$ to a basis for V:

$\{v_1, \dots, v_k, u_{k+1}, \dots, u_n\}$ (where $n = \dim V$)

Now apply G-S to this basis to obtain an orthogonal basis:

(Notice that $v_1 \dots v_k$ stay the same)

$\{v_1 \dots v_k, v_{k+1} \dots v_n\}$

By an earlier proposition (9H), u can be written as

$$u = \frac{\langle u, v_1\rangle}{\|v_1\|^2} v_1 + \dots + \frac{\langle u, v_k\rangle}{\|v_k\|^2} v_k + \frac{\langle u, v_{k+1}\rangle}{\|v_{k+1}\|^2} v_{k+1} + \dots + \frac{\langle u, v_n\rangle}{\|v_n\|^2} v_n$$

$= w + p$

Claim $p \in W^\perp$

pf We want to show that $\langle p, x\rangle = 0$ for all $x \in W$

Let $x \in W$. Then $x = \alpha_1 v_1 + \dots + \alpha_k v_k$

Then $\langle p, x\rangle = \left\langle \alpha_1 v_1 + \dots + \alpha_k v_k, \frac{\langle u, v_{k+1}\rangle}{\|v_{k+1}\|^2} v_{k+1} + \dots + \frac{\langle u, v_n\rangle}{\|v_n\|^2} v_n \right\rangle$

$= 0$ (exercise) $\square$ claim

$\square$ Prop

Ch 10: Orthogonal Matrices

Recall that A^t means "transpose of A."

so if $A = (a_{ij})$ then $A^t = (a_{ji})$

ex If $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix}$ then $A^t = \begin{pmatrix} a & e & i \\ b & f & j \\ c & g & k \\ d & h & l \end{pmatrix}$

Exercise
1. $(cA)^t = cA^t$ for any scalar c
2. $(A+B)^t = A^t + B^t$
3. $(AB)^t = B^t A^t$
4. If u, v are vectors in $\mathbb{R}^n$, then $u \cdot v = v^t u$

$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (v_1\ v_2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$u \cdot v = v^t u$
$\parallel$
$v \cdot u = u^t v$

Thus if A is invertible,

$\Rightarrow A A^{-1} = I$

$\Rightarrow (A A^{-1})^t = (I)^t$

$\Rightarrow (A^{-1})^t A^t = I$

$\Rightarrow (A^t)^{-1} = (A^{-1})^t$

This motivates the following definition

Defn A square matrix is called orthogonal if $A^{-1} = A^t$