Def A square matrix A is <u>orthogonal</u> if $A^{-1} = A^t$

Example ① $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

A is the standard matrix for the counterclockwise rotation by θ in $\mathbb{R}^2$

Thus $A^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A^t$

② $A = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}$

Theorem Suppose A is a real $n\times n$ matrix. Then A is orthogonal
$\Leftrightarrow$ The rows of A form an orthonormal basis for $\mathbb{R}^n$
$\Leftrightarrow$ The columns of A form an orthonormal basis for $\mathbb{R}^n$

Proof Let $r_1 \dots r_n$ be the rows of A.
We want to find out when $A^{-1} = A^t$ or $AA^t = I$

$$AA^t = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \begin{pmatrix} r_1 & \cdots & r_n \end{pmatrix} = \begin{pmatrix} r_1\cdot r_1 & r_1\cdot r_2 & \cdots & r_1\cdot r_n \\ r_2\cdot r_1 & r_2\cdot r_2 & \cdots & r_2\cdot r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_n\cdot r_1 & r_n\cdot r_2 & \cdots & r_n\cdot r_n \end{pmatrix}$$

So this is I iff $\begin{cases} r_i\cdot r_i = 1 & \text{for all } i \\ \text{and } r_i\cdot r_j = 0 & \text{if } i\neq j \end{cases}$

iff $\begin{cases} \|r_i\| = 1 \text{ for all } i \\ \text{and the rows are pair-wise orthogonal} \end{cases}$

iff the rows form an orthonormal basis.

(Proof for colums is similar) $\square$

<u>Thm</u> Suppose A is a real $n \times n$ matrix. then TFAE:

(a) A is orthogonal

(b) $\forall x \in \mathbb{R}^n$, $\|Ax\| = \|x\|$

(c) $\forall u, v \in \mathbb{R}^n$, $(Au)\cdot(Av) = u \cdot v$

<u>Proof</u> $(a \Rightarrow b)$ Suppose A is orthogonal. Let $x \in \mathbb{R}^n$

Then $\|Ax\|^2 = (Ax)\cdot(Ax) = (Ax)^t(Ax) = x^t A^t A x$

$$= x^t A^{-1} A x$$
$$= x^t x$$
$$= x \cdot x = \|x\|^2$$

$(b \Rightarrow c)$ Suppose $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$.

Let $u, v \in \mathbb{R}^n$

Then $(Au)\cdot(Av) = \frac{1}{4}\|Au + Av\|^2 - \frac{1}{4}\|Au - Av\|^2$

$$= \frac{1}{4}\|A(u+v)\|^2 - \frac{1}{4}\|A(u-v)\|^2$$

Exercise 9.10

$$\overset{\text{By Assumption}}{=} \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$$
$$= u \cdot v$$

$(c \Rightarrow a)$ Suppose $(Au)\cdot(Av) = u \cdot v$ for all $u, v \in \mathbb{R}^n$

Then $u \cdot v = (Au)\cdot(Av) = (Av)^t(Au) = v^t A^t A u = (A^t A u)\cdot v$

$\Rightarrow (Iu)\cdot v = (A^t A u)\cdot v$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow (A^t A u)\cdot v - (Iu)\cdot v = 0$

$\Rightarrow (A^t A u - Iu)\cdot v = 0$

$\Rightarrow [(A^t A - I)u]\cdot v = 0$

Since this is true for <u>all</u> u and v, let $v = (A^t A - I)u$

$\Rightarrow [(A^t A - I)u]\cdot[(A^t A - I)u] = 0$

$u \cdot u = 0$ iff $u = 0$

$\Rightarrow (A^t A - I)u = 0$

Since this is true for <u>all</u> u, then $A^t A - I = 0$

$$\Rightarrow A^t A = I \quad \square$$

<u>Theorem</u> Suppose $\mathcal{B} = \{u_1, \ldots u_n\}$ and $\mathcal{C} = \{v_1, \ldots v_n\}$ are orthonormal bases for a RIPS V.

Then if P is the transition matrix from $\mathcal{C}$ to $\mathcal{B}$, P is orthogonal.

Proof Let $u \in V$. Then there are unique scalars $\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n \in \mathbb{R}$ so that

$$u = \beta_1 u_1 + \dots + \beta_n u_n$$
$$= \gamma_1 v_1 + \dots + \gamma_n v_n$$

Then $\|u\|^2 = \langle u, u\rangle = \langle \beta_1 u_1 + \dots + \beta_n u_n, \beta_1 u_1 + \dots + \beta_n u_n\rangle$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_i \beta_j \boxed{\langle u_i, u_j\rangle} \quad \begin{matrix} = 0 \text{ if } i \neq j \\ = 1 \text{ if } i = j \end{matrix}$$

$$= \sum_{i=1}^{n} \beta_i^2$$

$$= (\beta_1, \dots \beta_n) \cdot (\beta_1, \dots \beta_n)$$

Similarly $\|u\|^2 = (\gamma_1, \dots, \gamma_n) \cdot (\gamma_1, \dots \gamma_n)$

From now on, consider $\| \ \|$ as the standard Euclidean norm.

$$\| [u]_{\mathcal{B}} \| = \| [u]_{\mathcal{C}} \| \quad \text{for all } u \in V$$

$$\Rightarrow \| P[u]_{\mathcal{C}} \| = \| [u]_{\mathcal{C}} \| \quad \text{for all } u \in V$$

Previous $\Rightarrow \|Px\| = \|x\|$ for all $x \in \mathbb{R}^n$

Thm. $\Leftrightarrow$ P is orthogonal

$\square$

Orthogonal Diagonalization

Defn (a) We say A is diagonalizable if we can write $P^{-1}AP = D$ where D is a diagonal matrix.

(In this case, the diagonal entries of D are the eigenvalues of A, including multiplicity. Also, the columns of P are the corresponding eigenvectors)

(b) We say A is orthogonally diagonalizable if we can write $P^{-1}AP = D$ where D is diagonal AND P is orthogonal.

Theorem A is orthogonal diagonalizable $\Leftrightarrow$ A is symmetric ($A^t = A$)

Proof ($\Leftarrow$) Omitted

($\Leftarrow$) Suppose $P^{-1}AP = D$ where D is diagonal + P is orthogonal

Then $A = PDP^{-1} = PDP^t = PD^tP^t$

So $A^t = (PD^tP^t)^t = (P^t)^t(D^t)^tP^t = PDP^t = PDP^{-1} = A$ $\square$

Corollary Suppose A is a real symmetric matrix.
Then all of the eigenvalues of A are real.

So How do we orthogonally diagonalize a symmetric matrix A?

1. Find eigenvalues & eigenvectors
2. Gram-Schmidt each eigenspace
3. Normalize each eigenvector.

But will P be orthogonal? Yes:

Theorem If A is symmetric and if u_1, u_2 are eigenvectors of A corresponding to the eigenvalues λ_1 and λ_2 ($\lambda_1 \neq \lambda_2$), Then $u_1 \cdot u_2 = 0$.

Proof $(Au_1) \cdot u_2 = u_2^t A u_1 = \underbrace{u_2^t A^t}_{} u_1 = u_1 \cdot \underbrace{(Au_2)}_{}$

Thus

$$(\lambda_1 u_1) \cdot u_2 = (Au_1) \cdot u_2 = u_1 \cdot (Au_2) = u_1 \cdot (\lambda_2 u_2)$$

$$\lambda_1 (u_1 \cdot u_2) \qquad\qquad \lambda_2 (u_1 \cdot u_2)$$

Thus $\lambda_1 (u_1 \cdot u_2) = \lambda_2 (u_1 \cdot u_2)$

$\Rightarrow (\lambda_1 - \lambda_2)(u_1 \cdot u_2) = 0$

since $\lambda_1 \neq \lambda_2$, $u_1 \cdot u_2 = 0$ $\square$

Example Orthogonally diagonalize $A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{pmatrix}$

We can calculate that $\chi(A) = -(\lambda - 7)(\lambda - 1)^2$

So the eigenvalues are $\lambda = 7$ + $\lambda = 1$ (mult 2)

Then $(1,2,1)$ is an eigenvector for $\lambda = 7$

and $(1,0,-1), (2,-1,0)$ are eigenvectors for $\lambda = 1$

Then $P^{-1}AP = D$ where $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & \underbrace{1 & 2}_{} \\ 2 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

Gram-Schmidt

$\Downarrow$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$\Downarrow$ normalize columns

$$P = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Oddtown

Near Chicago, there is a village called Eventown (pop. 100). In Eventown, the villagers like to form clubs. There are laws in Eventown, however, about how they are allowed to form clubs:

1. No two clubs can have the exact same list of members.
2. Each club has an even # of members
3. Each pair of clubs shares an even # of members.

At most

How many clubs can there be in Eventown?

<u>Theorem</u> Eventown can have at most 2^{50} clubs.

We can get 2^{50} clubs by pairing up the residents, and taking all possible combinations of the "married couples".

In the nearby village of Oddtown (pop 100), the residents also like to form clubs. In Oddtown, rules 1, 3 are the same.

However, they have rule 2a Each club has an odd # of members.

How many clubs can Oddtown have? No more than 100